

QUANTUM DYNAMICAL BOUNDS FOR ERGODIC POTENTIALS WITH UNDERLYING DYNAMICS OF ZERO TOPOLOGICAL ENTROPY

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ABSTRACT. In this paper we obtain upper quantum dynamical bounds as a corollary of positive Lyapunov exponent for Schrödinger operators $H_{f,\theta}u(n) = u(n+1) + u(n-1) + \phi(f^n\theta)u(n)$, where $\phi : \mathcal{M} \rightarrow \mathbb{R}$ is a piecewise Hölder function on a compact Riemannian manifold \mathcal{M} , and $f : \mathcal{M} \rightarrow \mathcal{M}$ is a uniquely ergodic volume preserving map with zero topological entropy. As corollaries we obtain localization-type statements for shifts and skew-shifts on higher dimensional tori with arithmetic conditions on the parameters. These are the first localization-type results with precise arithmetic conditions for multi-frequency quasiperiodic and skew-shift potentials.

1. INTRODUCTION

Positive Lyapunov exponents are generally viewed as a signature of localization. While it is known that they can coexist even with almost ballistic transport [24] [9], vanishing of certain dynamical exponents has been identified as a reasonable expected consequence of hyperbolicity of the corresponding transfer-matrix cocycle. Results in this direction were obtained in [7] [8] for one-frequency trigonometric polynomials, and recently in [15], for one-frequency quasiperiodic potentials under very mild assumptions on regularity of the sampling function. In this paper we identify a general property responsible for positive Lyapunov exponents implying vanishing of the dynamical quantities in the rather general case of underlying dynamics defined by volume preserving maps of Riemannian manifolds with zero topological entropy, and under very minimal regularity assumptions. This work presents the first localization-type results that hold in such generality. We expect that positive topological entropy should also lead to vanishing of the dynamical quantities for a.e. (but not every!) phase, but this should be approached by completely different methods and will be explored in a future work.

Our general results allow us, in particular, to obtain localization-type statements for potentials defined by shifts and skew-shifts of higher-dimensional tori. Pure point spectrum with exponentially decaying eigenfunctions has been obtained for a.e. multi-frequency shifts in the regime of positive Lyapunov exponents in [3] and for the skew-shift on \mathbb{T}^2 with a perturbative condition in [4], both very delicate results. While bounds on transport exponents are certainly weaker than dynamical localization that often (albeit not always [17]) accompanies pure point spectrum [5], we note that pure point spectrum can be destroyed by generic rank one perturbations [10] while vanishing of the transport exponents is robust in this respect. Finally, our results are the first ones for both of these families that hold under purely arithmetic conditions and the first non-perturbative ones for the skew-shift.

Let (\mathcal{M}, g) be a d -dimensional compact (smooth) Riemannian manifold with a metric g . Let Vol_g be its Riemannian volume density (see (2.1)). Let f be a uniquely ergodic volume preserving map on \mathcal{M} , which means Vol_g is its unique invariant probability measure. We will study the dynamical properties of the Schrödinger operator acting on $l^2(\mathbb{Z})$:

$$(1.1) \quad H_{f,\theta}u(n) = u(n+1) + u(n-1) + \phi(f^n\theta)u(n).$$

where $\theta \in \mathcal{M}$ is the phase.

The time dependent Schrödinger equation

$$i\partial_t u = H_\theta u,$$

leads to a unitary dynamical evolution

$$u(t) = e^{-itH_\theta} u(0).$$

Under the time evolution, the wavepacket will in general spread out with time. For operators with absolutely continuous spectrum, scattering theory leads to a good understanding of the quantum dynamics. In this paper we will study the spreading of the wavepacket under positive Lyapunov exponent assumption, which automatically implies the absence of absolutely continuous spectrum.

Let $e^{-itH_\theta} \delta_0$ be the time evolution with the localized initial state δ_0 . Let

$$a_\theta(n, t) = |\langle e^{-itH_\theta} \delta_0, \delta_n \rangle|^2.$$

$a_\theta(n, t)$ describes the probability of finding the wavepacket at site n at time t . We denote the p -th moment of $a_\theta(n, t)$ by

$$\langle |X|_\theta^p(t) \rangle = \sum_n (1 + |n|)^p a_\theta(n, t).$$

Dynamical localization is defined as boundedness of $\langle |X|_\theta^p(t) \rangle$ in time t . This implies purely point spectrum, therefore for general operators with positive Lyapunov exponent such a strong control of the wavepacket is not possible. Thus we need to define proper transport exponents which describe the rate of the spreading of the wavepacket. For $p > 0$ define the upper and lower transport exponents

$$\beta_\theta^+(p) = \limsup_{t \rightarrow \infty} \frac{\ln \langle |X|_\theta^p(t) \rangle}{p \ln t}; \quad \beta_\theta^-(p) = \liminf_{t \rightarrow \infty} \frac{\ln \langle |X|_\theta^p(t) \rangle}{p \ln t}.$$

Obtaining upper bounds for the two transport exponents above implies a power-law control of the spreading rate of the entire wavepacket.

It is also interesting to consider a portion of the wavepacket. For a nonnegative function $A(t)$ of time, let

$$\langle A(t) \rangle_T = \frac{2}{T} \int_0^\infty e^{-2t/T} A(t) dt$$

be its time average. Set

$$P_{\theta, T}(L) = \sum_{|n| \leq L} \langle a_\theta(n, t) \rangle_T.$$

Roughly speaking, $P_{\theta, T}(T^a) > \tau$ means that, in average, over time T , a portion of the wavepacket stays inside a box of size T^a . Let us introduce two other scaling exponents:

$$\begin{aligned} \overline{\xi}_\theta &= \lim_{\tau \rightarrow 0} \limsup_{T \rightarrow \infty} \frac{\ln \inf \{L | P_{\theta, T}(L) + P_{f\theta, T}(L) > \tau\}}{\ln T} \\ \underline{\xi}_\theta &= \lim_{\tau \rightarrow 0} \liminf_{T \rightarrow \infty} \frac{\ln \inf \{L | P_{\theta, T}(L) + P_{f\theta, T}(L) > \tau\}}{\ln T} \end{aligned}$$

The vanishing of β^\pm and $\overline{\xi}, \underline{\xi}$ can be viewed as localization-type statements. For $\mathcal{M} = \mathbb{T}$ the one-dimensional torus, $f : \theta \rightarrow \theta + \alpha$ the irrational rotation, the Lebesgue measure m is the unique invariant probability measure of f . It was first proved in [7], [8] that in this case for ϕ being a trigonometric polynomial, under the assumption of positive Lyapunov exponent, $\beta_\theta^+(p) = 0$ for all $p > 0$, all θ and Diophantine α ; $\beta_\theta^- = 0$ for all $p > 0$, all θ and all α . It was recently proved in [15] that under very mild restrictions on regularity of the potential, under the assumption of positivity and continuity of the Lyapunov exponent, $\beta_\theta^+(p) = 0$ for all $p > 0$, all θ and Diophantine α ; $\beta_\theta^-(p) = 0$ for all $p > 0$, all θ and all α . It was also proved in [15] that for piecewise Hölder

function, under the assumption of positive Lyapunov exponent, $\overline{\xi}_\theta = 0$ for a.e. θ and Diophantine α , $\underline{\xi}_\theta = 0$ for a.e. θ and all α .

Remark 1.1. The two Diophantine sets of α are different between [7], [8] and [15]. They are both full measure sets, but [15] covers a slightly thinner set of frequencies because of the need to handle potentials with weaker regularity.

In this paper we consider d -dimensional compact Riemannian manifold \mathcal{M} and uniquely ergodic volume preserving map f . We consider maps with the following volume scaling property. For $1 \leq l \leq d$, let $\Sigma(l)$ be the set of C^∞ mappings $\sigma : Q^l \rightarrow \mathcal{M}$ where Q^l is the l -dimensional unit cube. Let $\text{Vol}_{g,l}(\sigma)$ be the induced l -dimensional volume of the image of σ in \mathcal{M} counted with multiplicity, i.e. if σ is not one-to-one, and the image of one part coincides with that from another part, then we will count the set as many times as it is covered. For $n = 1, 2, \dots$ and $1 \leq l \leq d$, let

$$(1.2) \quad V_l(f) = \sup_{\sigma \in \Sigma(l)} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{Vol}_{g,l}(f^n \sigma) \quad \text{and} \quad V(f) = \max_l V_l(f).$$

Volume preserving f always satisfies $V_d(f) = V_d(f^{-1}) = 0$. Here we need to make an extra assumption that $V(f) = V(f^{-1}) = 0$. It is known that for smooth invertible map f , $V(f) = V(f^{-1})$ is equal to the *topological entropy* of f [27], thus our class of maps includes all smooth maps with zero topological entropy. In particular, it includes both the irrational rotation and the skew-shift.

For such maps we will assume that f has a bounded discrepancy.

Let $J_N(\theta) = J(\theta, f\theta, \dots, f^{N-1}\theta)$ (see (2.16)) be the isotropic discrepancy function of the sequence $\{f^n \theta\}_{n=0}^{N-1}$. For $\delta > 0$, we will say f has *strongly δ -bounded isotropic discrepancy* if $J_N(\theta) \leq |N|^{-\delta}$ uniformly in θ for $|N| > N_0$; f has *weakly δ -bounded isotropic discrepancy* if there exists a sequence $\{N_j\}$ such that $J_{N_j}(\theta) \leq |N_j|^{-\delta}$ uniformly in θ . It turns out many concrete dynamical systems feature these properties. We will show in Lemmas 3.6 - 3.8 that the following holds.

- Shifts of higher dimensional tori, $f : \theta \rightarrow \theta + \alpha$, has strongly bounded isotropic discrepancy for Diophantine α ;
- Skew-shift $f : (y_1, y_2, \dots, y_d) \rightarrow (y_1 + \alpha, y_2 + y_1, \dots, y_d + y_{d-1})$, has strongly bounded isotropic discrepancy for Diophantine α , and weakly bounded isotropic discrepancy for Liouvillean α .

Under the assumption of boundedness of discrepancy and scaling property of f , we are ready to formulate the following two abstract results.

Let μ_θ be the spectral measure of H_θ corresponding to δ_0 . Let $N = \int_{\mathcal{M}} \mu_\theta \, d\text{Vol}_g$ be the integrated density of states. Let $L(E)$ be the Lyapunov exponent, see (2.6).

Theorem 1.1. *Let ϕ be a piecewise Hölder function. Suppose $L(E)$ is positive on a Borel subset U with $N(U) > 0$. Suppose f is a uniquely ergodic volume preserving map satisfying $V(f) = V(f^{-1}) = 0$. We have*

- *If for some $\delta > 0$, f has weakly δ -bounded isotropic discrepancy, then $\underline{\xi}_\theta = 0$ for Vol_g -a.e. $\theta \in \mathcal{M}$;*
- *If for some $\delta > 0$, f has strongly δ -bounded isotropic discrepancy, then $\overline{\xi}_\theta = 0$ for Vol_g -a.e. $\theta \in \mathcal{M}$.*

Remark 1.2. The full measure set of θ appearing in Theorem 1.1 is precisely the set $\{\theta : \mu_\theta + \mu_{f\theta}(U) > 0\}$.

Theorem 1.2. *Under the assumption of Theorem 1.1, assume also $L(E)$ is continuous in E and $L(E) > 0$ for every $E \in \mathbb{R}$. We have*

- *If for some $\delta > 0$, f has weakly δ -bounded isotropic discrepancy, then $\beta_\theta^-(p) = 0$ for all $\theta \in \mathcal{M}$ and $p > 0$;*

- If for some $\delta > 0$, f has strongly δ -bounded isotropic discrepancy, then $\beta_\theta^+(p) = 0$ for all $\theta \in \mathcal{M}$ and $p > 0$.

Remark 1.3. Strongly δ -bounded isotropic discrepancy is essential for vanishing of $\bar{\xi}$ and $\beta_\theta^+(p)$, see Remarks 1.5 and 1.7. However, it is not yet clear whether weakly δ -bounded isotropic discrepancy (or any condition at all other than mere positivity of the Lyapunov exponent) is essential for vanishing of the ξ or of β_θ^- .

Theorems 1.1, 1.2 extend the results of [7, 8, 15] from irrational rotations of the circle to general uniquely ergodic maps of compact Riemannian manifolds with zero topological entropy and bounded discrepancy. One key to achieving such generality is a new argument that does not rely on harmonic analysis/ approximation by trigonometric polynomials.

By [6], $\beta_\theta^-(p) \geq p \dim_H(\mu_\theta)$ where $\dim_H(\mu)$ is the Hausdorff dimension of μ . Thus as a consequence of $\beta_\theta^-(p) = 0$ we have the following

Corollary 1.1. *Under the assumption of Theorem 1.2, $\dim_H(\mu_\theta) = 0$ for all $\theta \in \mathcal{M}$.*

Remark 1.4. The point here is that we obtain zero Hausdorff dimension of the spectral measure for *all* rather than a.e. $\theta \in \mathcal{M}$ (the latter is known for general ergodic potentials [26]). The statement for all θ has only been known for irrational rotations of \mathbb{T}^1 (proved for trigonometric polynomials in [14], and follows easily for piecewise functions from the results of [15]).

The following Theorems 1.3 - 1.6 are all corollaries of our abstract results. Theorems 1.7 and 1.8 depend on a somewhat different technique (bypassing the discrepancy considerations), which allows us to cover more frequencies in case of the shift of \mathbb{T}^2 . To our knowledge, Theorems 1.3 - 1.8 are the first arithmetic localization-type results.

Let us introduce the Diophantine condition and weak Diophantine condition on \mathbb{T}^d :

$$DC(\tau) = \cup_{c>0} DC(c, \tau) = \cup_{c>0} \{(\alpha_1, \dots, \alpha_d) \mid \|\langle \vec{h}, \alpha \rangle\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{c}{r(\vec{h})^\tau} \text{ for any } \vec{h} \neq \vec{0}\}$$

where $r(\vec{h}) = \prod_{i=1}^d \max(|h_i|, 1)$. It is well known that when $\tau > 1$, $DC(\tau)$ is a full measure set.

$$WDC(\tau) = \cup_{c>0} WDC(c, \tau) = \cup_{c>0} \{(\alpha_1, \dots, \alpha_d) \mid \max\{\|h\alpha_i\|_{\mathbb{R}/\mathbb{Z}}\} \geq \frac{c}{|h|^\tau} \text{ for any } h \neq 0\}, h \in \mathbb{Z}.$$

It is well known that when $\tau > \frac{1}{d}$, $WDC(\tau)$ is a full measure set.

Theorem 1.1 reduces vanishing of (upper or lower) ξ_θ to bounds on the isotropic discrepancy. As corollaries, we obtain

Theorem 1.3. *Let f be an irrational shift on \mathbb{T}^d . For piecewise Hölder ϕ , suppose $L(E)$ is positive on a Borel subset U with $N(U) > 0$. Then if $\alpha \in DC(\tau) \subset \mathbb{T}^d$, $\tau > 1$, we have $\bar{\xi}_\theta = 0$ for a.e. $\theta \in \mathbb{T}^d$.*

Remark 1.5. The Diophantine condition is essential for the vanishing of $\bar{\xi}$ [18].

Theorem 1.4. *Let f be a skew-shift. For piecewise Hölder ϕ , suppose $L(E)$ is positive on a Borel subset U with $N(U) > 0$. Then*

- for all irrational α , $\bar{\xi}_{\vec{y}} = 0$ for a.e. $\vec{y} \in \mathbb{T}^d$,
- if $\alpha \in DC(\tau)$ for some $\tau > 1$, $\bar{\xi}_{\vec{y}} = 0$ for a.e. $\vec{y} \in \mathbb{T}^d$.

Remark 1.6. The full measure set appearing in Theorems 1.3 and 1.4 is precisely the set $\{\theta : \mu_\theta + \mu_{f\theta}(U) > 0\}$.

Similarly, for systems with continuous Lyapunov exponent, Theorem 1.2 reduces vanishing of $\beta_\theta^\pm(p)$ to the same discrepancy bounds, and we obtain

Theorem 1.5. *Under the assumption of Theorem 1.3, assume in addition that $L(E)$ is continuous in E and $L(E) > 0$ for every $E \in \mathbb{R}$. Then if $\alpha \in DC(\tau) \subset \mathbb{T}^d$, $\beta_\theta^+(p) = 0$ for all $\theta \in \mathbb{T}^d$, $p > 0$.*

Corollary 1.2. *Under the assumption of Theorem 1.5, if $\alpha \in DC(\tau)$, $\dim_H(\mu_\theta) = 0$ for all $\theta \in \mathbb{T}^d$.*

Remark 1.7. The Diophantine condition is essential for $\beta^+ = 0$ [18].

Theorem 1.6. *Under the assumption of Theorem 1.4, assume in addition that $L(E)$ is continuous in E and $L(E) > 0$ for every $E \in \mathbb{R}$. Then*

- *for all irrational α , $\beta_{\vec{y}}^-(p) = 0$ for all $\vec{y} \in \mathbb{T}^d$, $p > 0$,*
- *if $\alpha \in DC(\tau)$ for some $\tau > 1$, $\beta_{\vec{y}}^+(p) = 0$ for all $\vec{y} \in \mathbb{T}^d$, $p > 0$.*

Corollary 1.3. *Under the assumption of Theorem 1.6, for all irrational α , $\dim_H(\mu_{\vec{y}}) = 0$ for all $\vec{y} \in \mathbb{T}^d$.*

Finally, for the case of the irrational shift \mathbb{T}^2 we can make two more delicate statements, using a different technique to obtain arithmetic estimates.

Theorem 1.7. *Let f be an irrational shift on \mathbb{T}^2 . For piecewise Hölder ϕ , suppose $L(E)$ is positive on a Borel subset U with $N(U) > 0$. Then if $\alpha = (\alpha_1, \alpha_2) \in \cup_{\tau > 1} WDC(\tau)$, we have $\underline{\xi}_\theta = 0$ for a.e. $\theta \in \mathbb{T}^2$.*

Remark 1.8. The full measure set appearing in Theorem 1.7 is precisely the set $\{\theta : \mu_\theta + \mu_{f\theta}(U) > 0\}$.

Theorem 1.8. *Under the assumption of Theorem 1.7, assume in addition that $L(E)$ is continuous in E and $L(E) > 0$ for every $E \in \mathbb{R}$. Then if $\alpha = (\alpha_1, \alpha_2) \in \cup_{\tau > 1} WDC(\tau)$, we have $\beta_\theta^-(p) = 0$ for all $\theta \in \mathbb{T}^2$, $p > 0$.*

Corollary 1.4. *Under the assumption of Theorem 1.8, if $\alpha \in \cup_{\tau > 1} WDC(\tau)$, we have $\dim_H(\mu_\theta) = 0$ for all $\theta \in \mathbb{T}^2$.*

The most technically complex part of the paper consists in obtaining arithmetic estimates on covering of the torus by the trajectory of a small ball in a polynomial (in the inverse radius) time, which we obtain by estimating the discrepancy in Theorems 1.3 - 1.6, and by the bounded remainder set technique in Theorems 1.7, 1.8. The discrepancy estimates are standard for the Diophantine shifts and are ideologically similar to the known results on equidistribution of $n^k \alpha$, for the case of higher dimensional Diophantine skew shifts. We still develop the proof for the Diophantine skew shift case in full detail because we did not find it in the literature and also because it serves as a good preparation to the Liouville higher dimensional skew shift, for which to the best of our knowledge, our estimates are new. We note that for the *Diophantine* skew shift of T^2 and shifts of T^d the results on the covering of the torus by a trajectory of a ball are shown in [1] by a completely different technique, through solving the cohomological equation. By the nature of the cohomological equation that technique is not extendable to the Liouville or weakly Diophantine case.

We organize this paper as follows: in section 2 we introduce some basic definitions. Some of them have been mentioned in the introduction but not in detail. In section 3 we will present some key lemmas and prove Theorems 1.1 - 1.8. In sections 4-8 we prove the key lemmas that are listed in section 3.

2. PREPARATION

2.1. Riemannian manifolds. Let \mathcal{M} be a d -dimensional compact Riemannian manifold with a Riemannian metric g .

Let K be a compact set in some coordinate patch (U, x^1, \dots, x^d) . We define the volume of K to be

$$\text{Vol}_g(K) := \int_{x(K)} \sqrt{|G \circ x^{-1}|} dx^1 \cdots dx^d,$$

where $G = \det g_{ij}$, $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ and $dx^1 \cdots dx^d$ is the Lebesgue measure on \mathbb{R}^d . This definition is free of choice of coordinate. If K is not contained in a single coordinate patch, one could apply partition of unity to define $\text{Vol}_g(K)$. More precisely, we pick an atlas $(U_\alpha, x_\alpha^1, \dots, x_\alpha^d)$ of \mathcal{M} and a partition of unity $\{\rho_\alpha\}$ subordinate to this atlas. Now we can set

$$\text{Vol}_g(K) = \sum_\alpha \int_{x_\alpha(K \cap U_\alpha)} (\rho_\alpha \sqrt{|G^\alpha|}) \circ (x^\alpha)^{-1} dx_\alpha^1 \cdots dx_\alpha^d.$$

The *Riemannian volume density* (see e.g. [25], section 3.4) on (\mathcal{M}, g) is

$$(2.1) \quad d\text{Vol}_g = \sum_\alpha (\rho_\alpha \sqrt{|G^\alpha|}) \circ (x^\alpha)^{-1} dx_\alpha^1 \cdots dx_\alpha^d.$$

In the above definition, we do not assume \mathcal{M} to be oriented. If \mathcal{M} is oriented, then the volume density is actually a positive n -form, called the volume form.

If $\varrho : [a, b] \rightarrow \mathcal{M}$ is a continuously differentiable curve in the Riemannian manifold \mathcal{M} , then we define its length $l(\varrho)$ by

$$l(\varrho) = \int_a^b \sqrt{g_{\varrho(t)}(\dot{\varrho}(t), \dot{\varrho}(t))} dt,$$

where $g_{\varrho(t)}$ is the inner product g at the point $\varrho(t)$. One could define the distance between any two point $x, y \in \mathcal{M}$ as follows

$$\text{dist}(x, y)$$

$$= \inf \{l(\varrho) : \varrho \text{ is a continuous, piecewise continuously differentiable curve connecting } x \text{ and } y\}.$$

With the definition of distance, *geodesics* in a Riemannian manifold are then the locally distance-minimizing paths.

Let $v \in T_x \mathcal{M}$ be a tangent vector to the manifold \mathcal{M} at x . Then there is a unique geodesic ϱ_v satisfying $\varrho_v(0) = x$ with initial tangent vector $\dot{\varrho}_v(0) = v$. The corresponding *exponential map* is defined by $\exp_x(v) = \varrho_v(1)$.

Let $B_r(x) = \{y \in \mathcal{M} : \text{dist}(x, y) < r\}$ be a *geodesic ball* centered at $x \in \mathcal{M}$ with radius r . It is known that $B_r(x) = \exp_x(B(0, r))$ where $B(0, r) = \{v \in T_x \mathcal{M} : g_x(v, v) < r\}$.

Proposition 2.1. *There exists $r_g > 0$ so that for all $r < r_g$, there exist positive constants C_g and c_g which are independent of $x \in \mathcal{M}$ so that*

$$(2.2) \quad c_g r^d \leq \text{Vol}_g(B_r(x)) \leq C_g r^d \text{ for any } x \in \mathcal{M}.$$

Proof. We will discuss the proof briefly. We could identify the tangent space $T_x \mathcal{M}$ isometrically with \mathbb{R}^d . Now $\exp_x : \mathbb{R}^d \rightarrow \mathcal{M}$ is a diffeomorphism on some small ball $B_{\mathbb{R}^d}(0, r)$. On this ball, straight lines are mapped to length-minimizing geodesics ([11], Proposition 3.6), and thus Euclidean balls are mapped to geodesic balls of the same radius. Taking r smaller if necessary, we can assume the Jacobian of \exp_x is bounded away from 0 and ∞ on $B_{\mathbb{R}^d}(0, r)$, thus for $r < r_x$ we have that $c_{g_x} r^d \leq \text{Vol}_g(B_r(x)) \leq C_{g_x} r^d$. Since \mathcal{M} is a compact manifold, we could take r_x, c_{g_x}, C_{g_x} independent of $x \in \mathcal{M}$. \square

A subset C of \mathcal{M} is said to be a *geodesically convex set* if, given any two points in C , there is a minimizing geodesic contained within C that joins those two points.

The *convexity radius at a point* $x \in \mathcal{M}$ is the supremum (which may be $+\infty$) of $r_x \in \mathbb{R}$ such that for all $r < r_x$ the geodesic ball $B_{r_x}(x)$ is geodesically convex. The *convexity radius of* (\mathcal{M}, g) is the infimum over the points $x \in \mathcal{M}$ of the convexity radii at these points.

Proposition 2.2. [2] *For compact manifold \mathcal{M} , the convexity radius r'_g of (\mathcal{M}, g) is positive.*

This clearly implies that for any $x \in \mathcal{M}$, any $r < r'_g$, $B_r(x)$ is geodesically convex.

2.2. Piecewise Hölder functions. Let $L_\gamma(\mathcal{M})$ be the space of γ -Lipschitz functions on \mathcal{M} . For $\phi \in L_\gamma(\mathcal{M})$ define

$$(2.3) \quad \|\phi\|_{L_\gamma} = \|\phi\|_\infty + \sup_{\theta_1, \theta_2 \in \mathcal{M}} \frac{|\phi(\theta_1) - \phi(\theta_2)|}{\text{dist}(\theta_1, \theta_2)^\gamma}.$$

We say ϕ is piecewise Hölder if there exists $\gamma > 0$, positive integer K and $\{\phi_j\}_{j=1}^K \subset L_\gamma(\mathcal{M})$ so that

$$\phi(\theta) = \sum_{j=1}^K \chi_{S_j}(\theta) \phi_j(\theta)$$

where $\{S_j\}_{j=1}^K$ are sets with “good boundary”, namely $\{\partial S_j\}_{j=1}^K$ are $d-1$ dimensional smooth submanifolds of \mathcal{M} . Clearly the discontinuity set J_ϕ of ϕ is $\cup_{j=1}^K \partial S_j$, and

$$(2.4) \quad \text{Vol}_{g,d-1}(J_\phi) \leq \sum_{j=1}^K \text{Vol}_{g,d-1}(\partial S_j) < \infty.$$

Clearly for any two points θ_1, θ_2 so that $\text{dist}(\theta_i, J_\phi) \geq r$, if $\text{dist}(\theta_1, \theta_2) < r$ then we have

$$(2.5) \quad |\phi(\theta_1) - \phi(\theta_2)| \leq \text{dist}(\theta_1, \theta_2)^\gamma \sum_{j=1}^K \|\phi_j\|_{L_\gamma}.$$

2.3. Cocycles and Lyapunov exponent. We now introduce the Lyapunov exponent. For a given $z \in \mathbb{C}$, a formal solution u of $Hu = zu$ can be reconstructed using the transfer matrix

$$A(\theta, z) = \begin{pmatrix} z - \phi(\theta) & -1 \\ 1 & 0 \end{pmatrix}$$

via the equation

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = A(f^n \theta, z) \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix}$$

Indeed, let $A_k(\theta, z)$ be the product of consecutive transfer matrices:

$$A_k(\theta, z) = A(f^{k-1} \theta, z) \cdots A(f \theta, z) A(\theta, z) \quad \text{for } k > 0, \quad A_0(\theta, z) = I \quad \text{and} \\ A_k(\theta, z) = (A_{-k}(f^k \theta, z))^{-1} \quad \text{for } k < 0.$$

Then for any $k \in \mathbb{Z}$ we have the following relation

$$\begin{pmatrix} u(k) \\ u(k-1) \end{pmatrix} = A_k(\theta, z) \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix}.$$

We define the Lyapunov exponent

$$(2.6) \quad L(z) = \lim_k \frac{1}{k} \int_{\mathcal{M}} \ln \|A_k(\theta, z)\| \, d\text{Vol}_g(\theta) = \inf_k \frac{1}{k} \int_{\mathcal{M}} \ln \|A_k(\theta, z)\| \, d\text{Vol}_g(\theta).$$

Furthermore, $L(z) = \lim_k \frac{1}{k} \ln \|A_k(\theta, z)\|$ for Vol_g -a.e. $\theta \in \mathcal{M}$.

2.4. Spectral measure and integrated density of states. Let μ_θ be the spectral measure of H_θ corresponding to δ_0 defined by

$$\langle (H_\theta - z)^{-1} \delta_0, \delta_0 \rangle = \int_{\mathbb{R}} \frac{d\mu_\theta(x)}{x - z}.$$

Then clearly $\mu_{f\theta}$ is the spectral measure of H_θ corresponding to δ_1 . Let $N = \int_{\mathcal{M}} \mu_\theta \, d\text{Vol}_g(\theta)$ be the integrated density of states. Then $N = \int_{\mathcal{M}} \frac{\mu_\theta + \mu_{f\theta}}{2} \, d\text{Vol}_g(\theta)$, so $N(U) > 0$ for some set U implies $\frac{\mu_\theta + \mu_{f\theta}}{2}(U) > 0$ for Vol_g -a.e. $\theta \in \mathcal{M}$.

2.5. Rational approximation.

2.5.1. Single frequency. Let α be an irrational number and let $\{\frac{p_n}{q_n}\}$ be its continued fraction approximants. We have the following properties (see e.g. [19]):

$$(2.7) \quad \frac{1}{2q_{n+1}} \leq \|q_n \alpha\|_{\mathbb{T}} \leq \frac{1}{q_{n+1}}.$$

$$(2.8) \quad \|k\alpha\| > \|q_n \alpha\| \text{ for } q_n < k < q_{n+1}.$$

(1) If $\alpha \in DC(c, \tau)$ for some $c > 0$, we have

$$(2.9) \quad \|k\alpha\|_{\mathbb{T}} \geq \frac{c}{|k|^\tau} \text{ for any } k \neq 0.$$

In particular, combining (2.7) with (2.9) we have

$$(2.10) \quad cq_{n+1} \leq q_n^\tau.$$

(2) If $\alpha \notin DC(\tau)$, there exists a subsequence of the continued fraction approximants $\{\frac{p_{n_k}}{q_{n_k}}\}$ so that

$$(2.11) \quad q_{n_k+1} > q_{n_k}^\tau.$$

2.5.2. Multiple frequencies. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ be a set of irrational frequencies. Let $\{\frac{\vec{p}_n}{q_n}\}$ be its best simultaneous approximation with respect to the Euclidean norm on \mathbb{T}^d , namely,

$$\sum_{j=1}^d \|q_n \alpha_j\|_{\mathbb{T}}^2 < \sum_{j=1}^d \|k \alpha_j\|_{\mathbb{T}}^2 \text{ for any } |k| < q_n.$$

Clearly by the pigeonhole principle, we have

$$(2.12) \quad \sqrt{\sum_{j=1}^d \|q_n \alpha_j\|_{\mathbb{T}}^2} \leq \frac{2\Gamma(\frac{d}{2} + 1)^{\frac{1}{d}}}{\sqrt{\pi} q_{n+1}^{\frac{1}{d}}}.$$

We say that

(1) $\alpha \in DC(c, \tau)$, if

$$(2.13) \quad \|\langle \vec{k}, \alpha \rangle\|_{\mathbb{T}} \geq \frac{c}{r(\vec{k})^\tau} \text{ for any } \vec{k} \in \mathbb{Z}^d \setminus \{\vec{0}\}.$$

(2) $\alpha \in WDC(c, \tau)$, if

$$(2.14) \quad \max_{1 \leq j \leq d} \|k \alpha_j\|_{\mathbb{T}} \geq \frac{c}{|k|^\tau} \text{ for any } k \in \mathbb{Z} \setminus \{\vec{0}\}.$$

2.6. Discrepancy. Let $\vec{x}_1, \dots, \vec{x}_N \in \mathcal{M}$. For a subset C of \mathcal{M} , let $A(C; \{\vec{x}_n\})$ be the counting function

$$(2.15) \quad A(C; \{\vec{x}_n\}_{n=1}^N) = \sum_{n=1}^N \chi_C(\vec{x}_n)$$

The *isotropic discrepancy* $J_N(\{\vec{x}_n\}_{n=1}^N)$ is defined as

$$(2.16) \quad J_N(\{\vec{x}_n\}_{n=1}^N) = \sup_{C \in \mathcal{C}} \left| \frac{A(C; \{\vec{x}_n\}_{n=1}^N)}{N} - \text{Vol}_g(C) \right|,$$

where \mathcal{C} is the family of all geodesically convex subsets of \mathcal{M} .

For a point $\theta \in \mathcal{M}$, let $J_N(\theta) = J(\{f^n \theta\}_{n=0}^{N-1})$. We say a map $f : \mathcal{M} \rightarrow \mathcal{M}$ has *strongly δ -bounded isotropic discrepancy* if for some $N > N_0$, $J_N(\theta) \leq N^{-\delta}$ uniformly in $\theta \in \mathcal{M}$. We say f has *weakly δ -bounded isotropic discrepancy* if there is a subsequence $\{N_j\}$ such that $J_{N_j}(\theta) \leq N_j^{-\delta}$ uniformly in $\theta \in \mathcal{M}$.

If $\mathcal{M} = \mathbb{T}^d$ is the d -dimensional torus, we define the *discrepancy* $D_N(\{\vec{x}_n\}_{n=1}^N)$ as follows

$$(2.17) \quad D(\{\vec{x}_n\}_{n=1}^N) = \sup_{C \in \mathcal{J}} \left| \frac{A(C; \{\vec{x}_n\}_{n=1}^N)}{N} - m(C) \right|,$$

where \mathcal{J} is the family of subintervals C of the form $C = \{(\theta_1, \dots, \theta_d) \in \mathbb{T}^d : \beta_i \leq \theta_i < \kappa_i \text{ for } 1 \leq i \leq d\}$.

For a point $\theta \in \mathbb{T}^d$, let $D_N(\theta) = D(\{f^n \theta\}_{n=0}^{N-1})$. We say a map $f : \mathbb{T}^d \rightarrow \mathbb{T}^d$ has *strongly δ -bounded discrepancy* if for some $N > N_0$, $D_N(\theta) \leq N^{-\delta}$ uniformly in $\theta \in \mathbb{T}^d$. We will say f has *weakly δ -bounded discrepancy* if there is a subsequence $\{N_j\}$ such that $D_{N_j}(\theta) \leq N_j^{-\delta}$ uniformly in $\theta \in \mathbb{T}^d$.

When $\mathcal{M} = \mathbb{T}^d$, the isotropic discrepancy and discrepancy can be tightly controlled by each other:

Lemma 2.1. ([22], *Theorem 1.6 in Chapter 2*) *For any sequence $\{\vec{x}_n\}_{n=1}^N$ in \mathbb{T}^d , we have*

$$(2.18) \quad D_N(\{\vec{x}_n\}_{n=1}^N) \leq J_N(\{\vec{x}_n\}_{n=1}^N) \leq (4d\sqrt{d} + 1)D_N(\{\vec{x}_n\}_{n=1}^N)^{\frac{1}{d}}.$$

Therefore, by (2.18), when $\mathcal{M} = \mathbb{T}^d$,

Proposition 2.3. *f has strongly (weakly) δ -bounded isotropic discrepancy for some $\delta > 0$ if and only if f has strongly (weakly) $\tilde{\delta}$ -bounded discrepancy for some $\tilde{\delta} > 0$.*

In section 5 and 6 we are going to apply the following two inequalities to estimate the discrepancy from above.

Lemma 2.2. [21] [*Erdős-Turán-Koksma inequality*] *For any positive integer H_0 , we have*

$$(2.19) \quad D(\{\vec{x}_n\}_{n=1}^N) \leq C_d \left(\frac{1}{H_0} + \sum_{0 < |\vec{h}| \leq H_0} \frac{1}{r(\vec{h})} \left| \frac{1}{N} \sum_{n=1}^N e^{2\pi i \langle \vec{h}, \vec{x}_n \rangle} \right| \right)$$

where $|\vec{h}| = \max_{j=1}^d |h_j|$.

Lemma 2.3. (e.g. [22], *Lemma 3.1 in Chapter 1*) [*Van der Corput's Fundamental Inequality*] *For any integer $1 \leq H \leq N$, we have*

$$(2.20) \quad \left| \frac{1}{N} \sum_{n=1}^N u_n \right|^2 \leq \frac{N+H-1}{N^2 H} \sum_{n=1}^N |u_n|^2 + \frac{2(N+H-1)}{N^2 H^2} \sum_{k=1}^{H-1} (H-k) \text{Re} \sum_{n=1}^{N-k} u_n \overline{u_{n+k}}.$$

3. KEY LEMMAS AND PROOFS OF THEOREM 1.1 - 1.8

3.1. Covering \mathcal{M} with the orbit of a geodesic ball and proofs of Theorem 1.1, 1.7, 1.2 and 1.8.

Lemma 3.1. *Let ϕ be a piecewise Hölder function with $1 \geq \gamma > 0$. Suppose $L(E)$ is positive on a Borel subset U with $N(U) > 0$.*

- (1) *If there exists a sequence $r_k \rightarrow 0$ so that any geodesic ball in \mathcal{M} with radius r_k covers the whole \mathcal{M} in r_k^{-M} steps, then $\underline{\xi}_\theta = 0$ for Vol_g -a.e. $\theta \in \mathcal{M}$;*
- (2) *If for any small $r > 0$, any geodesic ball with radius r covers the whole \mathcal{M} in r^{-M} steps, then $\overline{\xi}_\theta = 0$ for Vol_g -a.e. $\theta \in \mathcal{M}$;*

Lemma 3.2. *Let ϕ be a piecewise Hölder function with $1 \geq \gamma > 0$. Suppose $L(E)$ is continuous in E and $L(E) > 0$ for every $E \in \mathbb{R}$.*

- (1) *If there exists a sequence $r_k \rightarrow 0$ so that any geodesic ball in \mathcal{M} with radius r_k covers the whole \mathcal{M} in r_k^{-M} steps, then $\beta_\theta^-(p) = 0$ for all $\theta \in \mathcal{M}$ and $p > 0$;*
- (2) *If for any small $r > 0$, any geodesic ball with radius r covers the whole \mathcal{M} in r^{-M} steps, then $\beta_\theta^+(p) = 0$ for all $\theta \in \mathcal{M}$ and $p > 0$.*

Lemmas 3.1 and 3.2 are key to our abstract argument. They are proved in section 4. The connection to bounded discrepancy comes in the following

Let r_g be as in Proposition 2.1 and r'_g as in Proposition 2.2.

Lemma 3.3. *If f has weakly δ -bounded isotropic discrepancy, then there exists $r_k \rightarrow 0$ as $k \rightarrow \infty$ such that any geodesic ball in \mathcal{M} with radius r_k will cover the whole \mathcal{M} in $r_k^{-\frac{2d}{\delta}}$ steps.*

Proof. There exists a sequence $\{N_k\}$ and $k_0 > 0$ such that for any $k > k_0$ we have $J_{N_k}(\{f^n\theta\}_{n=0}^{N_k-1}) \leq N_k^{-\delta}$. This means for any geodesically convex set $C \subset \mathcal{M}$, $\frac{\sum_{n=0}^{N_k-1} \chi_C(f^n\theta)}{N_k} - \text{Vol}_g(C) \geq -N_k^{-\delta}$ holds for all $\theta \in \mathcal{M}$. Thus if we take $r_k = N_k^{-\frac{\delta}{2d}} < \min(r_g, r'_g)$, then by Proposition 2.2, we know $B_{r_k}(\theta)$ is geodesically convex. By Proposition 2.1, $\text{Vol}_g(B_{r_k}(\theta)) \geq c_g r_k^d = c_g N_k^{-\frac{\delta}{2}} > N_k^{-\delta}$. Thus $\sum_{n=0}^{r_k^{-\frac{2d}{\delta}}-1} \chi_{B_{r_k}(\theta)}(f^n\theta) > 0$ for any $\theta \in \mathcal{M}$. \square

Lemma 3.4. *If f has strongly δ -bounded isotropic discrepancy, then for any $0 < r < \min(r_g, r'_g)$, any geodesic ball in \mathcal{M} with radius r will cover the whole \mathcal{M} in $r^{-\frac{2d}{\delta}}$ steps.*

Proof. There exists N_0 such that for any $N > N_0$ we have $J_N(\{f^n\theta\}_{n=0}^{N-1}) \leq N^{-\delta}$ for all $\theta \in \mathcal{M}$. This means for any $0 < r < \min(r_g, r'_g)$, any geodesic ball $B_r(\theta)$ (it is geodesically convex by Proposition 2.2) and $N = r^{-\frac{2d}{\delta}}$ we have $\frac{\sum_{n=0}^{N-1} \chi_{B_r(\theta)}(f^n\theta)}{N} - \text{Vol}_g(B_r(\theta)) \geq -r^{-2d}$. Since by Proposition 2.1, $\text{Vol}_g(B_r(\theta)) \geq c_g r^d > r^{2d}$, we have $\sum_{n=0}^{r^{-\frac{2d}{\delta}}-1} \chi_{B_r(\theta)}(f^n\theta) > 0$ for any $\theta \in \mathcal{M}$. \square

In the case of 2-dimensional irrational rotation, we also have

Lemma 3.5. *For any $(\alpha_1, \alpha_2) \in \cup_{\tau>1} WDC(\tau)$, there exists $r_k(\alpha_1, \alpha_2, \tau) \rightarrow 0$ as $k \rightarrow \infty$ such that any Euclidean ball with radius r_k covers the whole \mathbb{T}^2 in $r_k^{-800\tau^4}$ steps.*

Remark 3.1. This lemma will be proved in section 8.

We are now ready to complete the proof of the main Theorems.

Proof of Theorem 1.1. Combining Lemma 3.3, 3.4 with Lemma 3.1. \square

Proof of Theorem 1.7. Combining Lemma 3.5 with Lemma 3.1. \square

Proof of Theorem 1.2. Combining Lemma 3.3, 3.4 with Lemma 3.2. \square

Proof of Theorem 1.8. Combining Lemma 3.5 with Lemma 3.2. \square

3.2. Estimation of Discrepancy and proofs of Theorems 1.3, 1.5, 1.4 and 1.6. We have the following control of the discrepancies of irrational rotation and skew-shift.

Lemma 3.6. *If $\alpha \in DC(\tau)$, then for some constant $\delta > 0$, $D_N(\{\theta + n\alpha\}_{n=0}^{N-1}) \leq N^{-\delta}$ uniformly in $\theta \in \mathbb{T}^d$.*

Let $\vec{Y}_n = (y_1 + \binom{n}{1}\alpha, y_2 + \binom{n}{1}y_1 + \binom{n}{2}\alpha, \dots, y_d + \binom{n}{1}y_{d-1} + \dots + \binom{n}{d}\alpha) = f^n(y_1, \dots, y_d)$, where f is the skew shift.

Lemma 3.7. *If $\alpha \in DC(\tau)$, then for some constant $\delta > 0$, $D_N(\{\vec{Y}_n\}_{n=1}^N) \leq N^{-\delta}$ uniformly in $(y_1, \dots, y_d) \in \mathbb{T}^d$.*

Lemma 3.8. *If $\alpha \notin DC(d)$, then for some constant $\delta > 0$ there exists a sequence $\{N_j\}$ so that $D_{N_j}(\{\vec{Y}_n\}_{n=1}^{N_j}) \leq N_j^{-\delta}$ uniformly in $(y_1, \dots, y_d) \in \mathbb{T}^d$.*

Remark 3.2. Lemma 3.6 is standard. It's proof will be given in the appendix. The proofs of Lemma 3.7 and 3.8 will be given in section 6.

Proof of Theorem 1.3, 1.5. Follows from Lemma 3.6 and Theorems 1.1, 1.2. \square

Proof of Theorem 1.4, 1.6. Follows from Lemmas 3.7, 3.8 and Theorems 1.1, 1.2. \square

4. PROOFS OF LEMMAS 3.1 AND 3.2

4.1. Upper and lower bounds on transfer matrices. The following lemma on the uniform upper bound of transfer matrix is essentially from [15]. We have adapted it into the following form for convenience.

Lemma 4.1. ([15], Theorem 3.1) *Let ϕ be a function whose discontinuity set has measure 0 and f be a uniquely ergodic map on \mathcal{M} . Then*

4.1.1. *Let $L(E)$ be positive on a Borel set U and μ be a measure with $\mu(U) > 0$. Then for any $\zeta > 0$ there exists a number $D_\zeta > 0$, and for any $\epsilon > 0$ there exists a set $B_{\zeta, \epsilon}$ with $0 < \mu(B_{\zeta, \epsilon}) < \zeta$, and an integer $N_{\zeta, \epsilon}$ so that for any $E \in U \setminus B_{\zeta, \epsilon}$:*

- (1) $L(E) \geq D_\zeta$,
- (2) for $n > N_{\zeta, \epsilon}$, $|z - E| < e^{-4\epsilon n}$ and $\theta \in \mathcal{M}$, we have $\frac{1}{n} \ln \|A_n(\theta, z)\| < L(E) + \epsilon$.

4.1.2. *Furthermore, if $L(E)$ is continuous in E and U is a compact set, there exists $D > 0$ and for any $\epsilon > 0$ there exists an integer N_ϵ so that for any $E \in U$:*

- (1) $L(E) \geq D$
- (2) for $n > N_\epsilon$, $|z - E| < e^{-4\epsilon n}$ and $\theta \in \mathcal{M}$, we have $\frac{1}{n} \ln \|A_n(\theta, z)\| < L(E) + \epsilon$.

We are also able to formulate the following lower bound for the norm of transfer matrices.

Lemma 4.2. *Let ϕ be a piecewise Hölder function with $1 \geq \gamma > 0$ and f be a uniquely ergodic volume preserving map on \mathcal{M} with $V(f) = V(f^{-1}) = 0$. Then*

4.2.1. *Let $L(E)$ be positive on a Borel set U and μ be a measure with $\mu(U) > 0$. Then for any $\zeta, \epsilon > 0$, let D_ζ , $B_{\zeta, \epsilon}$ and $N_{\zeta, \epsilon}$ be defined as in 4.1.1.*

- (1) If there exists a sequence $r_k \rightarrow 0$ so that any geodesic ball in \mathcal{M} with radius r_k covers the whole \mathcal{M} in r_k^{-M} steps, then there exists a sequence $\{n_k(\epsilon)\}$ such that for $k > k_{\zeta, \epsilon}$, any $E \in U \setminus B_{\zeta, \epsilon}$, $|z - E| < e^{-4\epsilon n_k}$ and $\theta \in \mathcal{M}$ we have

$$\min_{\iota \in \{-1, 1\}} \max_{\iota j=0, \dots, e^{\frac{5M\epsilon}{\gamma} n_k}} \|A_{n_k}(f^j \theta, z)\| \geq e^{n_k(L(E)-3\epsilon)}.$$

- (2) If for any small $r > 0$, any geodesic ball with radius r covers the whole \mathcal{M} in r^{-M} steps, then for $n > N'_{\zeta, \epsilon}$, any $E \in U \setminus B_{\zeta, \epsilon}$, $|z - E| < e^{-4\epsilon n}$ and $\theta \in \mathcal{M}$ we have

$$\min_{\iota \in \{-1, 1\}} \max_{\iota j=0, \dots, e^{\frac{5M\epsilon}{\gamma} n}} \|A_n(f^j \theta, z)\| \geq e^{n(L(E)-3\epsilon)}.$$

4.2.2. Furthermore, if $L(E)$ is continuous in E and U is a compact set, let D be defined as in 4.1.2 and for any $\epsilon > 0$ let N_ϵ be defined as in 4.1.2. Then for any $E \in U$ we have $L(E) \geq D$ and for any $|z - E| < e^{-4\epsilon n}$ we have

- (1) if there exists a sequence $r_k \rightarrow 0$ so that any geodesic ball in \mathcal{M} with radius r_k covers the whole \mathcal{M} in r_k^{-M} steps, then there exists a sequence $\{n_k(\epsilon)\}$ such that for $k > k_\epsilon$ and any $\theta \in \mathcal{M}$,

$$\min_{\iota \in \{-1, 1\}} \max_{\iota j=0, \dots, e^{\frac{5M\epsilon}{\gamma} n_k}} \|A_{n_k}(f^j \theta, z)\| \geq e^{n_k(L(E)-3\epsilon)}.$$

- (2) if for any small $r > 0$, any geodesic ball with radius r covers the whole \mathcal{M} in r^{-M} steps, then for $n > N'_\epsilon$ and any $\theta \in \mathcal{M}$,

$$\min_{\iota \in \{-1, 1\}} \max_{\iota j=0, \dots, e^{\frac{5M\epsilon}{\gamma} n}} \|A_n(f^j \theta, z)\| \geq e^{n(L(E)-3\epsilon)}.$$

Proof of Lemma 4.2. We will focus on the proof of part (1) of 4.2.1. The other three proofs will be discussed briefly at the end of this section.

For any $E \in U \setminus B_{\zeta, \epsilon}$ and $n > N_{\zeta, \epsilon}$, by Lemma 4.1.1 we have $\frac{1}{n} \|A_n(\theta, E)\| < L(E) + \epsilon$. Since $\int_{\mathcal{M}} \frac{1}{n} \ln \|A_n(\theta, E)\| d\text{Vol}_g(\theta) \geq L(E)$, we have

$$(4.1) \quad \text{Vol}_g(M_{n, E, L(E), \epsilon}) := \text{Vol}_g(\{\theta \in \mathcal{M} : \frac{1}{n} \ln \|A_n(\theta, E)\| > L(E) - \epsilon\}) > \frac{1}{2}.$$

Now we take any $\theta \in M_{n, E, L(E), \epsilon}$ and $|z - E| < e^{-4\epsilon n}$. When $n > 2N_{\zeta, \epsilon} + 3$, by the standard telescoping we have,

$$\begin{aligned} \|A_n(\theta, z)\| &\geq \|A_n(\theta, E)\| - \|A_n(\theta, z) - A_n(\theta, E)\| \\ &\geq e^{n(L(E)-\epsilon)} - (n + 2(N_{\zeta, \epsilon} + 1)) \|A\|_{\infty}^{N_{\zeta, \epsilon}} e^{n(L(E)-3\epsilon)} \\ &> e^{n(L(E)-2\epsilon)} \end{aligned}$$

for large enough $n > N'_{\zeta, \epsilon}$. This means

$$(4.2) \quad M_{n, E, L(E), \epsilon} \subset M_{n, z, L(E), 2\epsilon}.$$

We know the discontinuity set of $\frac{1}{n} \ln \|A_n(\theta, z)\|$ is $J_n = \cup_{l=0}^{n-1} f^{-l}(J_\phi)$, where $J_\phi = \cup_{j=1}^K \partial S_j$ is defined in section 2.2. By our assumption (2.4) and the fact the $V_{d-1}(f^{-1}) = 0$ (by the definition (1.2) of $V(f^{-1})$). For n large enough, we have

$$(4.3) \quad \text{Vol}_{g, d-1}(J_n) \leq e^{n\epsilon} \text{Vol}_{g, d-1}(J_\phi),$$

note that the largeness depends only on f . Define

$$\tilde{M}_{n, z, L(E), 2\epsilon} = M_{n, z, L(E), 2\epsilon} \setminus \overline{F_{2e^{-5\epsilon n/\gamma}}(J_n)},$$

where a neighborhood is defined as

$$F_r(A) = \{\theta \in \mathcal{M} : \text{dist}(\theta, A) < r\}.$$

Then by (4.3),

$$\begin{aligned} \text{Vol}_g(\tilde{M}_{n,z,L(E),2\epsilon}) &\geq \text{Vol}_g(M_{n,z,L(E),2\epsilon}) - 4e^{-5\epsilon n/\gamma} \text{Vol}_{g,d-1}(J_n) \\ &\geq \text{Vol}_g(M_{n,z,L(E),2\epsilon}) - 4e^{-n(\frac{5\epsilon}{\gamma}-\epsilon)} \text{Vol}_{g,d-1}(J_\phi) > \frac{2}{5}. \end{aligned}$$

In particular, it is a non-empty set. Now we take any $\tilde{\theta} \in \tilde{M}_{n,z,L(E),2\epsilon}$ and $\theta \in B_{e^{-5\epsilon n/\gamma}}(\tilde{\theta})$. We have, by telescoping, (2.5) and the fact that $V_1(f) = 0$ (by the definition (1.2) of $V(f)$),

$$\begin{aligned} \|A_n(\theta, z)\| &\geq \|A_n(\tilde{\theta}, z)\| - \|A_n(\theta, z) - A_n(\tilde{\theta}, z)\| \\ &\geq e^{n(L(E)-2\epsilon)} - \left(\sum_{l=1}^K \|\phi_l\|_{L^\gamma}\right)(n + 2(N_{\zeta,\epsilon} + 1)\|A\|_\infty^{N_{\zeta,\epsilon}})e^{n(L(E)+\epsilon)} \max_{j=0,\dots,n-1} (\text{dist}(f^j\theta, f^j\tilde{\theta}))^\gamma \\ &\geq e^{n(L(E)-2\epsilon)} - \left(\sum_{l=1}^K \|\phi_l\|_{L^\gamma}\right)(\text{dist}(\theta, \tilde{\theta}))^\gamma (n + 2(N_{\zeta,\epsilon} + 1)\|A\|_\infty^{N_{\zeta,\epsilon}})e^{n(L(E)+\epsilon+\gamma\epsilon)} \\ &> e^{n(L(E)-3\epsilon)}. \end{aligned}$$

for $n > N''_{\zeta,\epsilon}$. This means

$$F_{e^{-5\epsilon n/\gamma}}(\tilde{M}_{n,z,L(E),2\epsilon}) \subset M_{n,z,L(E),3\epsilon}.$$

Hence for $E \in U \setminus B_{\zeta,\epsilon}$, $n > N''_{\zeta,\epsilon}$ and $|z - E| < e^{-4\epsilon n}$, $M_{n,z,L(E),3\epsilon}$ contains a geodesic ball with radius $e^{-\frac{5\epsilon}{\gamma}n}$. Then there exists a sequence $\{n_k(\epsilon)\}$ such that a geodesic ball with radius $e^{-\frac{5\epsilon}{\gamma}n_k} \sim r_k$ covers the whole \mathcal{M} in at most $e^{\frac{5M\epsilon}{\gamma}n_k}$ steps. Thus for $E \in U \setminus B_{\zeta,\epsilon}$, $k > k_{\zeta,\epsilon}$ so that $n_k(\epsilon) > N''_{\zeta,\epsilon}$, any $|z - E| < e^{-4\epsilon n_k}$ and any $\theta \in \mathbb{T}^d$ we have

$$\min_{\iota \in \{-1,1\}} \max_{\iota j=0,\dots,e^{\frac{5M\epsilon}{\gamma}n_k}} \|A_{n_k}(f^j\theta, z)\| > e^{n_k(L(E)-3\epsilon)}.$$

Remark 4.1. Notice that part (2) of Lemma 4.2.1 follows without taking a subsequence $\{n_k(\epsilon)\}$. Also, 4.2.2 follows without excluding the set $B_{\zeta,\epsilon}$.

□

4.2. Dynamical bounds on ξ_θ . The key to estimate ξ_θ is to apply the following lemma by Killip, Kiselev and Last.

Following [13], for $f : \mathbb{Z} \rightarrow H$ where H is a Banach space, the truncated l^2 norms in the positive and negative directions are defined by

$$\begin{aligned} \|f\|_L^2 &= \sum_{n=1}^{\lfloor L \rfloor} |f(n)|^2 + (L - \lfloor L \rfloor)|f(\lfloor L \rfloor + 1)|^2 \text{ for } L > 0 \\ \|f\|_L^2 &= \sum_{n=0}^{\lfloor L \rfloor + 1} |f(n)|^2 + (\lfloor L \rfloor + 1 - L)|f(\lfloor L \rfloor)|^2 \text{ for } L < 0 \end{aligned}$$

The truncated l^2 norm in both directions is defined by

$$\|f\|_{L_1, L_2}^2 = \sum_{n=-\lfloor L_1 \rfloor}^{\lfloor L_2 \rfloor} |f(n)|^2 + (L_1 - \lfloor L_1 \rfloor)|f(-\lfloor L_1 \rfloor - 1)|^2 + (L_2 - \lfloor L_2 \rfloor)|f(\lfloor L_2 \rfloor + 1)|^2 \text{ for } L_1, L_2 \geq 1.$$

With $A_\bullet(\theta, z)$ being a function on \mathbb{Z} , define $\tilde{L}_\epsilon^+(\theta, z) \in \mathbb{R}^+$ and $\tilde{L}_\epsilon^-(\theta, z) \in \mathbb{R}^-$ by requiring

$$\|A_\bullet(\theta, z)\|_{\tilde{L}_\epsilon^\pm(\theta, z)} = 2\|A(\theta, z)\|\epsilon^{-1}.$$

Lemma 4.3. ([20], Theorem 1.5) *Let H_θ be a Schrödinger operator and μ_θ be the spectral measure of H_θ and δ_0 . Let $T > 0$ and $L_1, L_2 > 2$, then*

$$(4.4) \quad \left\langle \frac{1}{2}(\|e^{-itH_\theta}\delta_0\|_{L_1, L_2}^2 + \|e^{-itH_\theta}\delta_1\|_{L_1, L_2}^2) \right\rangle_T > C \frac{\mu_\theta + \mu_{f\theta}}{2}(\{E : |\tilde{L}_{T^{-1}}^-| \leq L_1; \tilde{L}_{T^{-1}}^+ \leq L_2\})$$

where C is an universal constant¹.

This lemma directly implies $P_{\theta, T}(L) + P_{f\theta, T}(L) > C \frac{\mu_\theta + \mu_{f\theta}}{2}(\{E : \|A_\bullet(\theta, z)\|_{\pm L} > 2\|A(\theta, z)\|T\})$. The plan is to show that for any $\eta > 1$, any θ_0 satisfying $(\mu_{\theta_0} + \mu_{f\theta_0})(U) > 0$, we have $(\mu_{\theta_0} + \mu_{f\theta_0})(\{E : \|A_\bullet(\theta_0, z)\|_{\pm T} > T^\eta\}) \gtrsim (\mu_{\theta_0} + \mu_{f\theta_0})(U)$.

Proof of Lemma 3.1. We will prove part (1) in detail. Part (2) will be discussed briefly at the end of this proof.

Fix $\eta > 1$. Fix θ_0 such that $(\mu_{\theta_0} + \mu_{f\theta_0})(U) > 0$. Let $\zeta = \frac{1}{2}(\mu_{\theta_0} + \mu_{f\theta_0})(U)$, so a constant. Let $D = D_\zeta$ from Lemma 4.1. Let $\epsilon = \min(\frac{\gamma D}{40M\eta}, \frac{D}{6})$. Then by Lemmas 4.1, there exists a set B , $0 < |B| < \frac{1}{2}(\mu_{\theta_0} + \mu_{f\theta_0})(U)$, and a sequence $\{n_k\}$, s.t. $L(E) \geq D$ on $U \setminus B$ and for $E \in U \setminus B$, $k \geq k_0$, $|z - E| < e^{-4\epsilon n_k}$ and any $\theta \in \mathcal{M}$,

$$\min_{\iota \in \{-1, 1\}} \max_{\iota j = 0, \dots, e^{\frac{10M\epsilon}{\gamma} n_k}} \|A_{n_k}(f^j \theta, z)\| > e^{n_k(L(E) - 3\epsilon)}.$$

Using that $A_{s+t}(\theta, z) = A_t(f^s(\theta), z)A_s(\theta, z)$, this implies, by the condition on ϵ ,

$$\|A_\bullet(\theta, z)\|_{\pm e^{\frac{10M\epsilon}{\gamma} n_k}} > e^{\frac{n_k(L(E) - 3\epsilon)}{2}} \geq e^{\frac{10M\epsilon}{\gamma} n_k \eta}.$$

If we take $T_k = e^{\frac{10M\epsilon}{\gamma} n_k}$, then $U \setminus B \subset \{E : \|A_\bullet(\theta, E)\|_{\pm T_k} > T_k^\eta\}$ for any θ , in particular θ_0 . Then by (4.4),

$$P_{\theta_0, T_k^\eta}(T_k) + P_{f\theta_0, T_k^\eta}(T_k) \geq C \frac{\mu_{\theta_0} + \mu_{f\theta_0}}{2}(\{E : \|A_\bullet(\theta_0, E)\|_{\pm T_k} > T_k^\eta\}) \geq \tilde{C} \frac{\mu_{\theta_0} + \mu_{f\theta_0}}{2}(U).$$

This implies $\xi_\theta = 0$ for all $\theta \in \mathcal{M}$ such that $(\mu_\theta + \mu_{f\theta})(U) > 0$.

Remark 4.2. Using Lemmas 4.1.1 (2), 4.2.1 (2) instead of 4.1.1 (1), 4.2.1 (1), Part (2) can be proved without taking a subsequence n_k therefore the conclusion holds for all T large enough rather than a sequence T_k . \square

4.3. Bounds on β . The key to the bounds on β is to apply the following lemma by Damanik and Tcheremchansev.

Lemma 4.4. (Theorem 1 of [7] plus Corollary 1 of [8]) *Let H be the Schrödinger operator, with f real valued and bounded, and $K \geq 4$ such that $\sigma(H) \subset [-K + 1, K - 1]$. Suppose for all $\rho \in (0, 1)$ we have*

$$(4.5) \quad \int_{-K}^K \left(\min_{\iota \in \{-1, 1\}} \max_{1 \leq \iota n \leq T^\rho} \|A_n(E + \frac{i}{T})\|^2 \right)^{-1} dE = O(T^{-\eta}).$$

for any $\eta \geq 1$. Then $\beta^+(p) = 0$ for all $p > 0$. If (4.5) is satisfied for a sequence $T_k \rightarrow \infty$, then $\beta^-(p) = 0$ for all $p > 0$.

¹Here we formulate this Lemma for operators with potential $V(n) = \phi(f^n \theta)$. This covers arbitrary bounded potentials by taking f to be a corresponding subshift.

Proof of Lemma 3.2. We will prove part (1) in detail. A modification needed for part (2) is discussed briefly at the end of this proof.

It suffices to consider small $\rho \in (0, 1)$. Fix any $\rho \in (0, 1)$ small and $\eta \geq 1$. Assume $\sigma(H) \subset [-K + 1, K - 1]$. Since $L(E)$ is continuous in E on a compact set $[-K, K]$, we have $L(E) \geq D > 0$ on $[-K, K]$. Fix $\epsilon_\eta = \min(\frac{\rho\gamma D}{20M\eta}, \frac{D}{6})$. By Lemma 4.2.2 there exists a sequence $\{n_{\eta,k}\}$ such that for any $E \in [-K, K]$, $k > k_\eta$, any $|z - E| < e^{-4\epsilon_\eta n_{\eta,k}}$ and any $\theta \in \mathcal{M}$,

$$\min_{\iota \in \{-1, 1\}} \max_{j=0, \dots, e^{\frac{5M\epsilon_\eta}{\gamma} n_{\eta,k}}} \|A_{n_{\eta,k}}(f^j \theta, z)\| > e^{n_{\eta,k}(L(E) - 3\epsilon_\eta)}.$$

Thus

$$\min_{\iota \in \{-1, 1\}} \max_{j=0, \dots, e^{\frac{10M\epsilon_\eta}{\gamma} n_{\eta,k}}} \|A_j(\theta, z)\|^2 \geq e^{n_{\eta,k}(L(E) - 3\epsilon_\eta)} \geq e^{\frac{10M\epsilon_\eta}{\gamma\rho} n_{\eta,k}\eta}$$

holds for any $\theta \in \mathcal{M}$, any $E \in [-K, K]$ and $|z - E| < e^{-4\epsilon_\eta n_{\eta,k}}$. Now we take $T_{\eta,k} = e^{\frac{10M\epsilon_\eta}{\gamma\rho} n_{\eta,k}}$,

$$|E + \frac{i}{T_{\eta,k}} - E| = \frac{1}{T_{\eta,k}} < e^{-4\epsilon_\eta n_{\eta,k}}.$$

Thus

$$\min_{\iota \in \{-1, 1\}} \max_{j=0, \dots, T_{\eta,k}^\rho} \|A_j(\theta, E + \frac{i}{T_{\eta,k}})\|^2 \geq T_{\eta,k}^\eta$$

holds for any $E \in [-K, K]$. Therefore

$$\int_{-K}^K \left(\min_{\iota \in \{-1, 1\}} \max_{1 \leq \iota n \leq T_{\eta,k}^\rho} \|A_n(\theta, E + \frac{i}{T_{\eta,k}})\|^2 \right)^{-1} dE \leq 2KT_{\eta,k}^{-\eta}.$$

Now take a sequence $\{k_i\}$ such that $T_{1,k_1} < T_{2,k_2} < \dots$. Let $T_m = T_{m,k_m}$. Then

$$\int_{-K}^K \left(\min_{\iota \in \{-1, 1\}} \max_{1 \leq \iota n \leq T_m^\rho} \|A_n(\theta, E + \frac{i}{T_m})\|^2 \right)^{-1} dE \leq 2KT_m^{-m}.$$

By (4.5), we have $\beta_\theta^-(p) \leq \rho$ for all $\theta \in \mathcal{M}$, any $\rho \in (0, 1)$ and any $p > 0$, thus $\beta_\theta^-(p) = 0$ for all $\theta \in \mathcal{M}$ and any $p > 0$.

Remark 4.3. Using Lemmas 4.1.2 (2) and 4.2.2 (2), part (2) follows without taking a subsequence $\{n_{\eta,k}\}$. Therefore the conclusion holds for all T large rather than a sequence T_k . \square

5. SKEW-SHIFT. PROOF OF LEMMAS 3.7 AND 3.8

In this section, we obtain the discrepancy bounds for the skew shift. While the Diophantine case is likely known, we didn't find this in the literature. We thus present a detailed proof, especially since we build our proof for the Liouvilian case on some of the same considerations.

Skew-shift. Let $f: \mathbb{T}^d \rightarrow \mathbb{T}^d$ be defined as follows

$$f(y_1, y_2, \dots, y_d) = (y_1 + \alpha, y_2 + y_1, \dots, y_d + y_{d-1}).$$

Let $\vec{Y}_n = f^n(y_1, \dots, y_d)$, then

$$(5.1) \quad \vec{Y}_n = (y_1 + \binom{n}{1}\alpha, y_2 + \binom{n}{1}y_1 + \binom{n}{2}\alpha, \dots, y_d + \binom{n}{1}y_{d-1} + \dots + \binom{n}{d}\alpha),$$

where $\binom{n}{m} = 0$ if $n < m$.

5.1. Preparation. Combinatorial identities.

Lemma 5.1. *Let $r_t \in \mathbb{N}$ for $1 \leq t \leq s$, then we have*

$$(5.2) \quad \sum_{1 \leq t \leq s}^{l_t=0,1} (-1)^{s-\sum_{i=1}^s l_i} \binom{\sum_{t=1}^s l_t r_t}{s-1} = 0,$$

$$(5.3) \quad \sum_{1 \leq t \leq s}^{l_t=0,1} (-1)^{s-\sum_{i=1}^s l_i} \binom{\sum_{t=1}^s l_t r_t}{s} = \prod_{t=1}^s r_t.$$

Proof. Let us consider the coefficient C_a of x^a in the product $(1+x)^{r_1} \cdot (1+x)^{r_2} \cdots (1+x)^{r_s} = (1+x)^{\sum_{i=1}^s r_i}$. Let us denote

$$(5.4) \quad A^{(a)} = \{(\vec{j}_1, \vec{j}_2, \dots, \vec{j}_s), \text{ where } \vec{j}_t = (j_{t,1}, j_{t,2}, \dots, j_{t,r_t}), j_{t,k} \in \{0,1\} \mid \sum_{t=1}^s \sum_{k=1}^{r_t} j_{t,k} = a\}$$

Each element in $A^{(a)}$ corresponds to one way of choosing 1 or x in each term of the product $(1+x)^{r_1} \cdot (1+x)^{r_2} \cdots (1+x)^{r_s}$ in order to get x^a , where $j_{t,k} = 0$ means we choose 1 out of the k -th $1+x$ from $(1+x)^{r_t}$, and $j_{t,k} = 1$ means we choose x instead of 1. Thus the capacity of $A^{(a)}$, denoted by $|A^{(a)}|$, is equal to $C_a = \binom{\sum_{t=1}^s r_t}{a}$. Let us further denote

$$(5.5) \quad A_t^{(a)} = A^{(a)} \cap \{\vec{j}_t = \vec{0}\}$$

For $a = s-1$, since it is impossible to obtain x^{s-1} with $\vec{j}_t \neq \vec{0}$ for any $1 \leq t \leq s$, we have

$$(5.6) \quad A^{(s-1)} \setminus (\cup_{t=1}^s A_t^{(s-1)}) = \emptyset.$$

For $a = s$,

$$(5.7) \quad A^{(s)} \setminus (\cup_{t=1}^s A_t^{(s)}) = D,$$

where

$$(5.8) \quad D = \{(\vec{j}_1, \vec{j}_2, \dots, \vec{j}_s) \mid \sum_{k=1}^{r_t} j_{t,k} = 1 \text{ for } 1 \leq t \leq s\}.$$

Clearly,

$$(5.9) \quad |\cup_{t=1}^s A_t^{(a)}| = \sum_{i=1}^s (-1)^{i-1} \sum_{1 \leq t_1 < t_2 < \dots < t_i \leq s} |\cap_{l=1}^i A_{t_l}^{(a)}|,$$

in which

$$(5.10) \quad \sum_{1 \leq t_1 < t_2 < \dots < t_i \leq s} |\cap_{l=1}^i A_{t_l}^{(a)}| = \sum_{\sum_{t=1}^s l_t = s-i}^{l_t=0,1} \binom{\sum_{t=1}^s l_t r_t}{a}.$$

Thus

$$(5.11) \quad \begin{aligned} |A^{(a)} \setminus (\cup_{t=1}^s A_t^{(a)})| &= \binom{\sum_{t=1}^s r_t}{a} + \sum_{i=1}^s (-1)^i \sum_{\sum_{t=1}^s l_t = s-i}^{l_t=0,1} \binom{\sum_{t=1}^s l_t r_t}{a}, \\ &= \sum_{1 \leq t \leq s}^{l_t=0,1} (-1)^{s-\sum_{i=1}^s l_i} \binom{\sum_{t=1}^s l_t r_t}{a}. \end{aligned}$$

For $a = s-1$, (5.2) follows directly from (5.6) and (5.11). For $a = s$, (5.3) follows from (5.7), (5.11) and the fact that $|D| = \prod_{t=1}^s r_t$. \square

5.2. Diophantine α . Proof of Lemma 3.7. For $\alpha \in DC(\tau)$, we take integers

$$(5.12) \quad H_j \sim N^{\frac{2^j}{(2^d-1)(\tau+\epsilon)}} \text{ for } 0 \leq j \leq d-1.$$

By Lemma 2.2,

$$(5.13) \quad \begin{aligned} D(\vec{Y}_1, \dots, \vec{Y}_N) &\leq C_d \left(\frac{1}{H_0} + \sum_{0 < |\vec{h}| \leq H_0} \frac{1}{r(\vec{h})} \left| \frac{1}{N} \sum_{n=1}^N e^{2\pi i \langle \vec{h}, \vec{Y}_n \rangle} \right| \right) \\ &= C_d \left(\frac{1}{H_0} + \sum_{0 < |\vec{h}| \leq H_0} \frac{1}{r(\vec{h})} \left| \frac{1}{N} \sum_{n=1}^N u_n^{(0)} \right| \right), \end{aligned}$$

where

$$(5.14) \quad u_n^{(0)} = \exp \left\{ 2\pi i \sum_{j=1}^d (h_j \alpha + \sum_{r=1}^{d-j} h_{j+r} y_r) \binom{n}{j} \right\}.$$

For $1 \leq s \leq d-2$, let

$$(5.15) \quad u_{k_1, \dots, k_s, n}^{(s)} = \exp \left\{ 2\pi i \sum_{j=s+1}^d (h_j \alpha + \sum_{r=1}^{d-j} h_{j+r} y_r) \sum_{1 \leq t \leq s}^{l_t=0,1} (-1)^{s-\sum_{t=1}^s l_t} \binom{n + \sum_{t=1}^s l_t k_t}{j} \right\}$$

Then by Lemma 2.3,

$$(5.16) \quad \begin{aligned} &\left| \frac{1}{N - \sum_{t=1}^s k_t} \sum_{n=1}^{N - \sum_{t=1}^s k_t} u_{k_1, \dots, k_s, n}^{(s)} \right|^2 \\ &\lesssim \frac{1}{H_{s+1}} + \frac{1}{(N - \sum_{t=1}^s k_t) H_{s+1}^2} \sum_{k_{s+1}=1}^{H_{s+1}} (H_{s+1} - k_{s+1}) \left| \sum_{n=1}^{N - \sum_{t=1}^{s+1} k_t} u_{k_1, \dots, k_s, n}^{(s)} \overline{u_{k_1, \dots, k_s, n+k_{s+1}}^{(s)}} \right|. \end{aligned}$$

Here

$$\begin{aligned}
& \left| \sum_{n=1}^{N-\sum_{t=1}^{s+1} k_t} u_{k_1, \dots, k_s, n}^{(s)} \overline{u_{k_1, \dots, k_s, n+k_{s+1}}^{(s)}} \right| \\
&= \left| \sum_{n=1}^{N-\sum_{t=1}^{s+1} k_t} \exp \left\{ 2\pi i \sum_{j=s+1}^d (h_j \alpha + \sum_{r=1}^{d-j} h_{j+r} y_r) \sum_{1 \leq t \leq s}^{l_t=0,1} (-1)^{s-\sum_{t=1}^s l_t} \left(\binom{n + \sum_{t=1}^s l_t k_t}{j} - \binom{n + k_{s+1} + \sum_{t=1}^s l_t k_t}{j} \right) \right\} \right| \\
&= \left| \sum_{n=1}^{N-\sum_{t=1}^{s+1} k_t} \exp \left\{ 2\pi i \sum_{j=s+1}^d (h_j \alpha + \sum_{r=1}^{d-j} h_{j+r} y_r) \sum_{1 \leq t \leq s+1}^{l_t=0,1} (-1)^{s+1-\sum_{t=1}^{s+1} l_t} \binom{n + \sum_{t=1}^{s+1} l_t k_t}{j} \right\} \right| \\
&= \left| \sum_{n=1}^{N-\sum_{t=1}^{s+1} k_t} \exp \left\{ 2\pi i \sum_{j=s+1}^d (h_j \alpha + \sum_{r=1}^{d-j} h_{j+r} y_r) \sum_{0 \leq t \leq s+1}^{l_t=0,1} (-1)^{s+2-\sum_{t=0}^{s+1} l_t} \binom{l_0 n + \sum_{t=1}^{s+1} l_t k_t}{j} \right\} \right| \\
(5.17) \quad &= \left| \sum_{n=1}^{N-\sum_{t=1}^{s+1} k_t} \exp \left\{ 2\pi i \sum_{j=s+2}^d (h_j \alpha + \sum_{r=1}^{d-j} h_{j+r} y_r) \sum_{0 \leq t \leq s+1}^{l_t=0,1} (-1)^{s+2-\sum_{t=0}^{s+1} l_t} \binom{l_0 n + \sum_{t=1}^{s+1} l_t k_t}{j} \right\} \right| \\
&= \left| \sum_{n=1}^{N-\sum_{t=1}^{s+1} k_t} \exp \left\{ 2\pi i \sum_{j=s+2}^d (h_j \alpha + \sum_{r=1}^{d-j} h_{j+r} y_r) \sum_{1 \leq t \leq s+1}^{l_t=0,1} (-1)^{s+1-\sum_{t=1}^{s+1} l_t} \binom{n + \sum_{t=1}^{s+1} l_t k_t}{j} \right\} \right| \\
(5.18) \quad &= \left| \sum_{n=1}^{N-\sum_{t=1}^{s+1} k_t} u_{k_1, \dots, k_{s+1}, n}^{(s+1)} \right|.
\end{aligned}$$

Notice that in (5.17), we applied (5.3),

$$\exp \left\{ (h_{s+1} \alpha + \sum_{r=1}^{d-s-1} h_{s+1+r} y_r) \sum_{0 \leq t \leq s+1}^{l_t=0,1} (-1)^{s+2-\sum_{t=0}^{s+1} l_t} \binom{l_0 n + \sum_{t=1}^{s+1} l_t k_t}{s+1} \right\} = 1.$$

Combining (5.16) with (5.18), we get for any $0 \leq s \leq d-3$,

$$\begin{aligned}
(5.19) \quad & \left| \frac{1}{N - \sum_{t=1}^s k_s} \sum_{n=1}^{N-\sum_{t=1}^s k_t} u_{k_1, \dots, k_s, n}^{(s)} \right|^2 \\
& \leq \frac{1}{H_{s+1}} + \frac{1}{(N - \sum_{t=1}^s k_t) H_{s+1}^2} \sum_{k_{s+1}=1}^{H_{s+1}} (H_{s+1} - k_{s+1}) (N - \sum_{t=1}^{s+1} k_t) \left| \frac{1}{N - \sum_{t=1}^{s+1} k_t} \sum_{n=1}^{N-\sum_{t=1}^{s+1} k_t} u_{k_1, \dots, k_{s+1}, n}^{(s+1)} \right|.
\end{aligned}$$

By (5.16), for $s = d - 2$,

$$\begin{aligned}
 (5.20) \quad & \left| \frac{1}{N - \sum_{l=1}^{d-2} k_l} \sum_{n=1}^{N - \sum_{l=1}^{d-2} k_l} u_{k_1, \dots, k_{d-2}, n}^{(d-2)} \right|^2 \\
 & \lesssim \frac{1}{H_{d-1}} + \frac{1}{(N - \sum_{l=1}^{d-2} k_l) H_{d-1}^2} \sum_{k_{d-1}=1}^{H_{d-1}} (H_{d-1} - k_{d-1}) \left| \sum_{n=1}^{N - \sum_{l=1}^{d-1} k_l} u_{k_1, \dots, k_{d-2}, n}^{(d-2)} \overline{u_{k_1, \dots, k_{d-2}, n+k_{d-1}}^{(d-2)}} \right| \\
 & \lesssim \frac{1}{H_{d-1}} + \frac{1}{(N - \sum_{l=1}^{d-2} k_l) H_{d-1}} \sum_{k_{d-1}=1}^{H_{d-1}} \left| \sum_{n=1}^{N - \sum_{l=1}^{d-1} k_l} u_{k_1, \dots, k_{d-2}, n}^{(d-2)} \overline{u_{k_1, \dots, k_{d-2}, n+k_{d-1}}^{(d-2)}} \right|,
 \end{aligned}$$

and

$$\begin{aligned}
 & \left| \sum_{n=1}^{N - \sum_{l=1}^{d-1} k_l} u_{k_1, \dots, k_{d-2}, n}^{(d-2)} \overline{u_{k_1, \dots, k_{d-2}, n+k_{d-1}}^{(d-2)}} \right| \\
 & = \left| \sum_{n=1}^{N - \sum_{l=1}^{d-1} k_l} \exp\{2\pi i h_d \alpha \sum_{1 \leq l \leq d-1}^{j_l=0,1} (-1)^{d-1-\sum_{l=1}^{d-1} j_l} \binom{n + \sum_{j=1}^{d-1} j_l k_l}{d}\} \right| \\
 & = \left| \sum_{n=1}^{N - \sum_{l=1}^{d-1} k_l} \exp\{2\pi i h_d \alpha \sum_{0 \leq l \leq d-1}^{j_l=0,1} (-1)^{d-\sum_{l=0}^{d-1} j_l} \binom{l_0 n + \sum_{j=1}^{d-1} j_l k_l}{d}\} \right| \\
 (5.21) \quad & = \left| \sum_{n=1}^{N - \sum_{l=1}^{d-1} k_l} \exp\{2\pi i h_d n \alpha \prod_{l=1}^{d-1} k_l\} \right| \\
 (5.22) \quad & \lesssim \frac{1}{\|h_d \alpha \prod_{l=1}^{d-1} k_l\|_{\mathbb{T}}},
 \end{aligned}$$

where in (5.21) we used (5.3).

Since $\alpha \in DC(\tau)$, by the property of Diophantine condition (2.9) and since $|h_i| \leq H_0$, $1 \leq k_i \leq H_i$ we have

$$(5.23) \quad \sum_{k_{d-1}=1}^{H_{d-1}} \frac{1}{\|h_d \alpha \prod_{l=1}^{d-1} k_l\|_{\mathbb{T}}} \leq \sum_{j=1}^{H_{d-1}} \frac{m^\tau \prod_{l=1}^{d-1} H_l^\tau}{j} \leq m^\tau H_{d-1}^{\tau+\epsilon} \prod_{l=1}^{d-2} H_l^\tau.$$

Thus combining (5.20), (5.22) with (5.23), we have

$$\left| \frac{1}{N - \sum_{l=1}^{d-2} k_l} \sum_{n=1}^{N - \sum_{l=1}^{d-2} k_l} u_{k_1, \dots, k_{d-2}, n}^{(d-2)} \right|^2 \lesssim \frac{1}{H_{d-1}} + \frac{m^\tau H_{d-1}^{\tau+\epsilon} \prod_{l=1}^{d-2} H_l^\tau}{H_{d-1} (N - \sum_{l=1}^{d-2} H_l)} \lesssim \frac{1}{H_{d-1}} = \frac{1}{H_{d-2}^2}.$$

Lemma 5.2. For any $\alpha \in \mathbb{T}$, if for any $1 \leq k_s \leq H_s$,

$$\left| \frac{1}{N - \sum_{l=1}^s k_l} \sum_{n=1}^{N - \sum_{l=1}^s k_l} u_{k_1, \dots, k_s, n}^{(s)} \right|^2 \lesssim \frac{1}{H_s^2},$$

then for any $0 \leq t \leq s - 1$, $1 \leq k_t \leq H_t$ we have

$$\left| \frac{1}{N - \sum_{l=1}^t k_l} \sum_{n=1}^{N - \sum_{l=1}^t k_l} u_{k_1, \dots, k_t, n}^{(t)} \right|^2 \lesssim \frac{1}{H_t^2}.$$

Proof. For $t = s - 1$, by (5.19),

$$\begin{aligned} & \left| \frac{1}{N - \sum_{l=1}^{s-1} k_l} \sum_{n=1}^{N - \sum_{l=1}^{s-1} k_l} u_{k_1, \dots, k_{s-1}, n}^{(s-1)} \right|^2 \\ & \lesssim \frac{1}{H_s} + \frac{1}{(N - \sum_{l=1}^{s-1} k_l) H_s^2} \sum_{k_s=1}^{H_s} (H_s - k_s) (N - \sum_{l=1}^s k_l) \left| \frac{\sum_{n=1}^{N - \sum_{l=1}^s k_l} u_{k_1, \dots, k_s, n}^{(s)}}{(N - \sum_{l=1}^s k_l)} \right| \\ & \lesssim \frac{1}{H_s} = \frac{1}{H_{s-1}^2}. \end{aligned}$$

Then we proceed by reverse induction. □

At the final step we obtain

$$\left| \frac{1}{N} \sum_{n=1}^N u_n^{(0)} \right|^2 \lesssim \frac{1}{H_0^2}$$

Plugging it into (5.13), we have

$$D(\vec{Y}_1, \dots, \vec{Y}_N) \lesssim \frac{1}{H_0} + \sum_{0 < |\vec{h}| \leq H_0} \frac{1}{r(\vec{h})} \frac{1}{H_0} \lesssim \frac{1}{H_0^{1-\epsilon}} \sim N^{-\frac{1-\epsilon}{(2^d-1)(\tau+\epsilon)}}.$$

□

5.3. Liouvillean α , Proof of Lemma 3.8. For $\alpha \notin DC(d)$, by property (2.11), we could find a subsequence $\{\frac{p_n}{q_n}\}$ of the continued fraction approximants of α , so that $q_{n+1} > q_n^d$. In the following we will use q instead of q_n and \tilde{q} instead of q_{n+1} for simplicity. Here we would like to show $D_q(\vec{Y}_1, \dots, \vec{Y}_q) \leq q^{-\delta}$ for some $\delta > 0$. Take

$$(5.24) \quad H_j \sim q^{\frac{2^j}{2^d}} \text{ for } 0 \leq j \leq d-2 \quad \text{and} \quad H_{d-1} \sim q^{\frac{2^{d-1}(1+\epsilon)}{2^d}},$$

where $\epsilon > 0$ is small enough so that

$$(5.25) \quad \prod_{l=0}^{d-1} H_l = q^{\frac{2^{d-1} + 2^{d-1}\epsilon}{2^d}} < q.$$

Now by Lemma 2.2

$$(5.26) \quad D(\vec{Y}_1, \dots, \vec{Y}_q) \leq C_d \left(\frac{1}{H_0} + \sum_{0 < |\vec{h}| \leq H_0} \frac{1}{r(\vec{h})} \left| \frac{1}{q} \sum_{n=1}^q \exp\{2\pi i \sum_{j=1}^d (h_j \alpha + h_{j+1} y_1 + \dots + h_d y_{d-j})\} \binom{n}{j} \right| \right)$$

Consider the following difference

(5.27)

$$\begin{aligned}
& \frac{1}{q} \left| \sum_{n=1}^q \exp\left\{2\pi i \sum_{j=1}^d (h_j \alpha + h_{j+1} y_1 + \dots + h_d y_{d-j}) \binom{n}{j}\right\} - \sum_{n=1}^q \exp\left\{2\pi i \sum_{j=1}^d (h_j \frac{p}{q} + h_{j+1} y_1 + \dots + h_d y_{d-j}) \binom{n}{j}\right\} \right| \\
& \leq \frac{1}{q} \sum_{n=1}^q \left| \exp\left\{2\pi i \sum_{j=1}^d h_j \left(\alpha - \frac{p}{q}\right) \binom{n}{j}\right\} - 1 \right| \\
& \lesssim \frac{1}{q} \sum_{n=1}^q \sum_{j=1}^d \binom{n}{j} H_0 \left| \alpha - \frac{p}{q} \right| \\
& \lesssim \frac{H_0}{q},
\end{aligned}$$

where in the last step we use (2.7), $|\alpha - \frac{p}{q}| \leq \frac{1}{q\bar{q}} < \frac{1}{q^{d+1}}$.

Then combining (5.26) with (5.27), we have

$$(5.28) \quad D(\vec{Y}_1, \dots, \vec{Y}_q) \lesssim C_d \left(\frac{1}{H_0} + \sum_{0 < |\vec{h}| \leq H_0} \frac{1}{r(\vec{h})} \left| \frac{1}{q} \sum_{n=1}^q u_n^{(0)} \right| \right) + \frac{H_0}{q},$$

where $\tilde{u}_n^{(0)} = \exp\{2\pi i \sum_{j=1}^d (h_j \frac{p}{q} + h_{j+1} y_1 + \dots + h_d y_{d-j}) \binom{n}{j}\}$, that is $u_n^{(0)}$ as in (5.14) with α replaced with $\frac{p}{q}$. Thus with $\tilde{u}_{k_1, \dots, k_s, n}^{(s)}$ defined as in (5.15) with α replaced with $\frac{p}{q}$, similar to (5.20) and (5.21), we have

$$\begin{aligned}
& \left| \frac{1}{N - \sum_{l=1}^{d-2} k_l} \sum_{n=1}^{N - \sum_{l=1}^{d-2} k_l} \tilde{u}_{k_1, \dots, k_{d-2}, n}^{(d-2)} \right|^2 \\
(5.29) \quad & \lesssim \frac{1}{H_{d-1}} + \frac{1}{(N - \sum_{l=1}^{d-2} k_l) H_{d-1}} \sum_{k_{d-1}=1}^{H_{d-1}} \left| \sum_{n=1}^{N - \sum_{l=1}^{d-1} k_l} \tilde{u}_{k_1, \dots, k_{d-2}, n}^{(d-2)} \overline{\tilde{u}_{k_1, \dots, k_{d-2}, n+k_{d-1}}^{(d-2)}} \right|,
\end{aligned}$$

and

$$\begin{aligned}
& \left| \sum_{n=1}^{q - \sum_{l=1}^{d-1} k_l} \tilde{u}_{k_1, \dots, k_{d-2}, n}^{(d-2)} \overline{\tilde{u}_{k_1, \dots, k_{d-2}, n+k_{d-1}}^{(d-2)}} \right| \\
& = \left| \sum_{n=1}^{q - \sum_{l=1}^{d-1} k_l} \exp\left\{2\pi i h_d n \frac{p}{q} \prod_{l=1}^{d-1} k_l\right\} \right| \\
(5.30) \quad & \lesssim \frac{1}{\|h_d \frac{p}{q} \prod_{l=1}^{d-1} k_l\|_{\mathbb{R}/\mathbb{Z}}}.
\end{aligned}$$

Since $|h_d| \leq H_0$, $1 \leq k_i \leq H_i$ and (5.25), for any $1 \leq k \leq H_{d-1}$ we have $\|k h_d \frac{p}{q} \prod_{l=1}^{d-2} k_l\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{1}{q}$. Thus

$$(5.31) \quad \sum_{k_{d-1}=1}^{H_{d-1}} \frac{1}{\|h_d \frac{p}{q} \prod_{l=1}^{d-1} k_l\|_{\mathbb{R}/\mathbb{Z}}} \lesssim \sum_{j=1}^{H_{d-1}} \frac{q}{j} \leq q \ln H_{d-1}.$$

Then combining (5.29), (5.30) with (5.31), we get

$$(5.32) \quad \left| \frac{1}{q - \sum_{l=1}^{d-2} k_l} \sum_{n=1}^{q - \sum_{l=1}^{d-2} k_l} \tilde{u}_{k_1, \dots, k_{d-2}, n}^{(d-2)} \right|^2 \lesssim \frac{1}{H_{d-1}} + \frac{q \ln H_{d-1}}{(q - \sum_{l=1}^{d-2} H_l) H_{d-1}} \lesssim \frac{1}{H_{d-1}^{\frac{1}{1+\epsilon}}} = \frac{1}{H_{d-2}^2}.$$

By Lemma 5.2,

$$\left| \frac{1}{q} \sum_{n=1}^q \tilde{u}_n^{(0)} \right|^2 \lesssim \frac{1}{H_0}.$$

Plugging it into (5.28), we get

$$D(\vec{Y}_1, \dots, \vec{Y}_q) \lesssim \frac{1}{H_0} + \frac{(\log H_0)^d}{H_0} + \frac{H_0}{q} \lesssim \frac{1}{q^{\frac{1-\epsilon}{2^d}}}.$$

6. BOUNDED REMAINDER SETS

Most of the material covered in this section comes from [12]. We briefly discuss it here for completeness and readers' convenience. From now on we restrict our attention to irrational rotation on \mathbb{T}^d . For a measurable set $U \subset \mathbb{T}^d$, consider the function $A_N(U, \vec{x}) - N|U| := A(U, \{\vec{x} + n\alpha\}_{n=0}^{N-1}) - N|U| = \sum_{n=0}^{N-1} \chi_U(\vec{x} + n\alpha) - N|U|$. We will say U is a *bounded remainder set (BRS)* with respect to α if there exists a constant $C(U, \alpha) > 0$ such that $|A_N(U, \vec{x}) - N|U|| \leq C(U, \alpha)$ for any N and a.e. $\vec{x} \in \mathbb{T}^d$. We will call a measurable function g on \mathbb{T}^d a transfer function for U if its characteristic function satisfies

$$\chi_U(\vec{x}) - |U| = g(\vec{x}) - g(\vec{x} - \alpha) \quad \text{a.e.}$$

Obviously if g is a transfer function for U , then its Fourier coefficients satisfy

$$(6.1) \quad \hat{g}(\vec{m}) = \frac{\hat{\chi}_U(\vec{m})}{1 - e^{-2\pi i \langle \vec{m}, \alpha \rangle}}, \quad \vec{m} \neq 0.$$

Proposition 6.1. [12] *For a measurable set $U \subset \mathbb{T}^d$, the following are equivalent:*

- U is a bounded remainder set.
- U has a bounded transfer function g .

Theorems 6.2, 6.3 and Corollary 6.1 are presented in [12] without explicit bounds on the transfer functions. We present the proofs in order to extract the needed estimates.

Theorem 6.2. *Any interval $I \subset \mathbb{T}$ of length $0 < |q\alpha - p| < 1$ is a BRS with respect to α , furthermore its transfer function g satisfies $\|g\|_\infty \leq |q|$.*

Proof. Without loss of generality, we consider an interval $I = [0, \kappa]$, where $\kappa = q\alpha - p > 0$. Then

$$\begin{aligned} \chi_I(x) - |I| &= -\{x\} + \{x - \kappa\} \\ &= -\{x\} + \{x - q\alpha\} \\ &= (-\{x\} - \dots - \{x - (q-1)\alpha\}) + (\{x - \alpha\} + \dots + \{x - q\alpha\}) \\ &= g(x) - g(x - \alpha), \end{aligned}$$

where $g(x) = -\sum_{j=0}^{q-1} \{x - j\alpha\}$, $\|g\|_\infty \leq |q|$. □

Theorem 6.3. Let $\vec{v} = (v_1, v_2, \dots, v_d) = q\alpha - \vec{p} \in \mathbb{Z}\alpha + \mathbb{Z}^d$, $v \notin \mathbb{Z}^d$, and let $\Sigma \in \mathbb{T}^{d-1}$ be a BRS with respect to the vector $(\frac{v_1}{v_d}, \frac{v_2}{v_d}, \dots, \frac{v_{d-1}}{v_d})$ with transfer function h . Then the set

$$U = U(\Sigma, \vec{v}) = \{(\vec{x}, 0) + t\vec{v} : \vec{x} \in \Sigma, 0 \leq t < 1\},$$

is a BRS with respect to α , whose transfer function g satisfies $\|g\|_\infty \leq |q|(\|h\|_\infty + 1)$.

Proof. Let $\vec{v}_0 = (v_1, \dots, v_{d-1})$ be the vector in \mathbb{T}^{d-1} , which consists of the first $d-1$ entries of \vec{v} . First, we wish to find a bounded function \tilde{g} on \mathbb{T}^d satisfying the cohomological equation

$$\chi_U(\vec{x}, y) - |U| = \tilde{g}(\vec{x}, y) - \tilde{g}(\vec{x} - \vec{v}_0, y - v_d) \quad \text{for a.e. } (\vec{x}, y) \in \mathbb{T}^{d-1} \times \mathbb{T}.$$

This means the Fourier coefficients satisfy the equation

$$(6.2) \quad \hat{g}(\vec{m}, n)(1 - e^{-2\pi i(\langle \vec{m}, \vec{v}_0 \rangle + nv_d)}) = \int_0^{v_d} \int_{\Sigma + \frac{y}{v_d} \vec{v}_0} e^{-2\pi i(\langle \vec{m}, \vec{x} + \frac{y}{v_d} \vec{v}_0 \rangle)} d\vec{x} e^{-2\pi i ny} dy, \quad (\vec{m}, n) \neq (\vec{0}, 0).$$

Which implies

$$(6.3) \quad \hat{g}(\vec{m}, n) = \frac{\hat{\chi}_\Sigma(\vec{m})}{2\pi i(\langle \vec{m}, \vec{v}_0 \rangle / v_d + n)}, \quad (\vec{m}, n) \neq (\vec{0}, 0).$$

We know Σ is a BRS with respect to \vec{v}_0/v_d , by (6.1) its transfer function $h : \mathbb{T}^{d-1} \rightarrow \mathbb{R}$ satisfies

$$\hat{h}(\vec{m}) = \frac{\hat{\chi}_\Sigma(\vec{m})}{1 - e^{-2\pi i(\langle \vec{m}, \vec{v}_0 \rangle / v_d)}}, \quad \vec{m} \neq 0.$$

It is straightforward to check that the bounded function \tilde{g} defined by

$$\tilde{g}(\vec{x}, y) = h(\vec{x} - \frac{\vec{v}_0}{v_d} \{y\}) - |\Sigma| \cdot \{y\},$$

satisfies the cohomological equation (6.3). Hence \tilde{g} is a bounded transfer function for U with respect to \vec{v} .

Indeed, $\|\tilde{g}\|_\infty \leq \|h\|_\infty + 1$. Since $\vec{v} = q\alpha - \vec{p}$, letting $g(\vec{x}) = \tilde{g}(\vec{x}) + \tilde{g}(\vec{x} - \alpha) + \dots + \tilde{g}(\vec{x} - (q-1)\alpha)$ we have that U is a BRS with respect to α with bounded transfer function g satisfying $\|g\|_\infty \leq |q|\|\tilde{g}\|_\infty \leq |q|(\|h\|_\infty + 1)$. \square

The following corollary will be used several times in section 8.

Corollary 6.1. Let $U \subset \mathbb{T}^2$ be the parallelogram spanned by two vectors $m(\alpha_1, \alpha_2) - (l_1, l_2)$ and $(q\frac{m\alpha_1 - l_1}{m\alpha_2 - l_2} - p, 0)$, then U is a BRS with respect to (α_1, α_2) with transfer function g satisfying $\|g\|_\infty \leq |m|(|q| + 1) \leq 2|m q|$.

Proof. In this case $v = (v_1, v_2) = m(\alpha_1, \alpha_2) - (l_1, l_2) \in \mathbb{Z}\alpha + \mathbb{Z}^2$, $\Sigma = [0, q\frac{v_1}{v_2} - p] \times \{0\}$. We know the transfer function h of Σ with respect to v_1/v_2 satisfies $\|h\|_\infty \leq |q|$. Thus $\|g\|_\infty \leq |m|(|q| + 1) \leq 2|m q|$. \square

7. 2-DIMENSIONAL IRRATIONAL ROTATION WITH WEAK DIOPHANTINE FREQUENCIES

In this section we deal with 2-dimensional weakly Diophantine frequencies. Our goal is to prove Lemma 3.5.

Proof of Lemma 3.5. Assume $(\alpha_1, \alpha_2) \in WDC(c_0, \tau/4)$, for some $\tau > 4$ and $c_0 > 0$. We divide the discussion into two parts.

First, we introduce the coprime Diophantine condition:

$$(7.1) \quad PDC(\tau) = \cup_{c>0} PDC(c, \tau) = \cup_{c>0} \{(\alpha_1, \alpha_2) \mid \|\langle \vec{h}, \alpha \rangle\|_{\mathbb{T}} \geq \frac{c}{|\vec{h}|^\tau} \text{ for any } \gcd(h_1, h_2) = 1 \\ \text{or } h_1 h_2 = 0 \text{ but } \vec{h} \neq \vec{0}\}.$$

Obviously if $\alpha \in PDC(c, \tau)$, both α_1 and α_2 belong to $DC(c, \tau)$.

Case A. $(\alpha_1, \alpha_2) \in PDC(c_1, \tau)$ for some $c_1 > 0$.

Let's take the best simultaneous approximation $\{(\frac{l_{1,n}}{m_n}, \frac{l_{2,n}}{m_n})\}$ of (α_1, α_2) . They feature the following property.

Lemma 7.1. ([23], Theorem 3.5) *If $\{1, \alpha_1, \alpha_2\}$ is linearly independent over \mathbb{Q} , then there are infinitely many n_k such that*

$$\begin{vmatrix} m_{n_k} & l_{1,n_k} & l_{2,n_k} \\ m_{n_k+1} & l_{1,n_k+1} & l_{2,n_k+1} \\ m_{n_k+2} & l_{1,n_k+2} & l_{2,n_k+2} \end{vmatrix} \neq 0$$

Now we take $r_k > 0$ such that

$$(7.2) \quad m_{n_k} \leq \frac{4}{\pi} r_k^{-2} < m_{n_k+1}.$$

By (2.12), the choice of r_k guarantees that for $n \geq n_k$,

$$(7.3) \quad (m_n \alpha_1 - l_{1,n}, m_n \alpha_2 - l_{2,n}) \in B_{r_k}(0, 0),$$

where $B_r(x_1, x_2) := \{y = (y_1, y_2) \in \mathbb{T}^2 : \|y_1 - x_1\|_{\mathbb{T}}^2 + \|y_2 - x_2\|_{\mathbb{T}}^2 < r^2\}$. Let $\{\frac{p_{n,s}}{q_{n,s}}\}_{s=1}^\infty$ be the continued fraction approximants of $\frac{m_n \alpha_1 - l_{1,n}}{m_n \alpha_2 - l_{2,n}}$. For each n choose s_n such that

$$(7.4) \quad q_{n,s_n} \leq r_k^{-1} < q_{n,s_n+1}.$$

By (2.7), the choice of s_n guarantees that

$$(7.5) \quad (q_{n,s_n} \frac{m_n \alpha_1 - l_{1,n}}{m_n \alpha_2 - l_{2,n}} - p_{n,s_n}, 0) \in B_{r_k}(0, 0).$$

By (2.12) and (2.14) we have

$$(7.6) \quad \frac{c_0}{m_n^{\tau/4}} \leq \max\{|m_n \alpha_1 - l_{1,n}|, |m_n \alpha_2 - l_{2,n}|\} \leq \frac{2}{\sqrt{\pi} \sqrt{m_{n+1}}},$$

by (7.2) we have $m_{n_k} \leq \frac{4}{\pi} r_k^{-2}$, thus

$$(7.7) \quad \max(m_{n_k}, m_{n_k+1}, m_{n_k+2}) \leq C_{c_0, \tau} r_k^{-\frac{\tau^2}{2}}.$$

Case A.1: For some $n \in \{n_k, n_k + 1, n_k + 2\}$, we have $q_{n,s_n+1} \leq r_k^{-2\tau^4}$.

Let U be the parallelogram spanned by the two vectors $m_n(\alpha_1, \alpha_2) - (l_{1,n}, l_{2,n})$ and $(q_{n,s_n} \frac{m_n \alpha_1 - l_{1,n}}{m_n \alpha_2 - l_{2,n}} - p_{n,s_n}, 0)$. By (7.3) and (7.5), $U \subset B_{2r_k}(0, 0)$. Corollary 6.1 implies that $|\sum_{j=0}^{M-1} \chi_U(x + j\alpha_1, y + j\alpha_2) - M|U|| \leq 4|m_n q_{n,s_n}|$ for a.e. (x, y) . Thus as long as $M > \frac{4|m_n q_{n,s_n}|}{|U|}$, we should have $\cup_{j=0}^{M-1} U - (j\alpha_1, j\alpha_2)$ covers the whole \mathbb{T}^2 up to a measure zero set. Then

$$(7.8) \quad \mathbb{T}^2 \subseteq \cup_{j=0}^{M-1} B_{2r_k}(-j\alpha_1, -j\alpha_2) \text{ for } M > \frac{4|m_n q_{n,s_n}|}{|U|}.$$

Now we want to estimate $|U|$. Since $\alpha_2 \in DC(c_1, \tau)$, by (2.9) we have

$$|U| = |m_n \alpha_2 - l_{2,n}| \cdot |q_{n,s_n} \frac{m_n \alpha_1 - l_{1,n}}{m_n \alpha_2 - l_{2,n}} - p_{n,s_n}| \geq \frac{c_1}{|m_n|^\tau} \frac{1}{2q_{n,s_n+1}}.$$

Thus by (7.4) and (7.7),

$$\frac{4|m_n|q_{n,s_n}}{|S|} \leq \frac{8}{c_1} |m_n|^{1+\tau} q_{n,s_n} q_{n,s_n+1} \leq C_{c_0, c_1, \tau} r_k^{-3\tau^4}.$$

This means it takes $B_{2r_k}(0, 0)$ at most $C_{\alpha_1, \alpha_2, \tau} r_k^{-3\tau^4}$ steps to cover the whole \mathbb{T}^2 .

Case A.2. We will show now it is impossible to have $q_{n,s_n+1} > r_k^{-2\tau^4}$ for all $n \in \{n_k, n_k + 1, n_k + 2\}$. In this case by (2.7), (2.12) and (7.2), we have:

(7.9)

$$|q_{n,s_n} m_n \alpha_1 - p_{n,s_n} m_n \alpha_2 + M_n| = |m_n \alpha_2 - l_{2,n}| \cdot |q_{n,s_n} \frac{m_n \alpha_1 - l_{1,n}}{m_n \alpha_2 - l_{2,n}} - p_{n,s_n}| < \frac{2}{\sqrt{\pi} \sqrt{|m_{n+1}|} q_{n,s_n}} < r_k^{2\tau^4+1}$$

where $M_n = p_{n,s_n} l_{2,n} - q_{n,s_n} l_{1,n}$.

We have the following estimates on the upper bounds of p_{n,s_n} and M_n . Combining (2.9), (7.2), (7.4), (7.6) with (7.7),

$$(7.10) \quad |p_{n,s_n}| \leq q_{n,s_n} \left| \frac{m_n \alpha_1 - l_{1,n}}{m_n \alpha_2 - l_{2,n}} \right| + \frac{1}{q_{n,s_n+1}} \leq \frac{2q_{n,s_n} |m_n|^\tau}{c_1 \sqrt{\pi} \sqrt{|m_{n+1}|}} + r_k^{2\tau^4} \leq C_{c_0, c_1, \tau} r_k^{-\frac{\tau^3}{2}}.$$

By (7.9), (7.2), (7.7), (7.4) and (7.10),

$$(7.11) \quad |M_n| < |q_{n,s_n} m_n \alpha_1 - p_{n,s_n} m_n \alpha_2| + r_k^{2\tau^4} \leq C_{c_0, c_1, \tau} r_k^{-\tau^3}.$$

Case A.2.1. If $p_{n,s_n} = 0$ for some $n \in \{n_k, n_k + 1, n_k + 2\}$, then by (2.7), (2.12) and (7.1), (2.9), (7.2), (7.7), we have

$$r_k^{2\tau^4} > \frac{1}{q_{n,s_n+1}} \geq |q_{n,s_n} \frac{m_n \alpha_1 - l_{1,n}}{m_n \alpha_2 - l_{2,n}}| \geq \frac{c_1 \sqrt{\pi} \sqrt{|m_{n+1}|}}{2m_n^\tau} \geq C_{c_0, c_1, \tau} r_k^{\frac{\tau^3}{2}+1},$$

which is a contradiction.

Case A.2.2. If $M_n = 0$ for some $n \in \{n_k, n_k + 1, n_k + 2\}$, then by (7.9), (7.2), (7.10), and the fact that $(\alpha_1, \alpha_2) \in PDC(c_1, \tau)$, we have

$$r_k^{2\tau^4} > |m_n| |q_{n,s_n} \alpha_1 - p_{n,s_n} \alpha_2| \geq \frac{c_1 |m_n|}{\max(p_{n,s_n}, q_{n,s_n})^\tau} \geq C_{c_0, c_1, \tau} r_k^{\frac{\tau^4}{2}},$$

again a contradiction.

Case A.2.3. If $p_{n,s_n} \neq 0$ and $M_n \neq 0$ for any $n \in \{n_k, n_k + 1, n_k + 2\}$, then for any $i, j \in \{n_k, n_k + 1, n_k + 2\}$, we have:

$$(7.12) \quad \begin{aligned} & |(q_{i,s_i} m_i M_j - q_{j,s_j} m_j M_i) \alpha_1 - (p_{i,s_i} m_i M_j - p_{j,s_j} m_j M_i) \alpha_2| \\ & \leq |(q_{i,s_i} m_i \alpha_1 - p_{i,s_i} m_i \alpha_2 + M_i) M_j| + |(q_{j,s_j} m_j \alpha_1 - p_{j,s_j} m_j \alpha_2 + M_j) M_i| \\ & < (|M_i| + |M_j|) r_k^{2\tau^4}. \end{aligned}$$

Case A.2.3.1. $(q_{i,s_i}m_iM_j - q_{j,s_j}m_jM_i, p_{i,s_i}m_iM_j - p_{j,s_j}m_jM_i) \neq (0, 0)$ for some $i, j \in \{n_k, n_k + 1, n_k + 2\}$.

In this case let $h = \gcd(q_{i,s_i}m_iM_j - q_{j,s_j}m_jM_i, p_{i,s_i}m_iM_j - p_{j,s_j}m_jM_i)$ be the greatest common divisor of the two numbers if they are both nonzero, and $h = 1$ otherwise. Then by (7.12),

$$\left| \frac{q_{i,s_i}m_iM_j - q_{j,s_j}m_jM_i}{h}\alpha_1 - \frac{p_{i,s_i}m_iM_j - p_{j,s_j}m_jM_i}{h}\alpha_2 \right| < \frac{|M_i| + |M_j|}{h}r_k^{2\tau^4}.$$

However on one hand by (7.11),

$$\frac{|M_i| + |M_j|}{h}r_k^{2\tau^4} \leq (|M_i| + |M_j|)r_k^{2\tau^4} \leq C_{c_0, c_1, \tau}r_k^{2\tau^4 - \tau^3}.$$

On the other hand, by the fact that $(\alpha_1, \alpha_2) \in PDC(c_1, \tau)$ and (7.2), (7.7), (7.10), (7.11),

$$\begin{aligned} & \left| \frac{q_{i,s_i}m_iM_j - q_{j,s_j}m_jM_i}{h}\alpha_1 - \frac{p_{i,s_i}m_iM_j - p_{j,s_j}m_jM_i}{h}\alpha_2 \right| \\ & \geq \frac{c_1h^\tau}{|(q_{i,s_i}m_iM_j - q_{j,s_j}m_jM_i, p_{i,s_i}m_iM_j - p_{j,s_j}m_jM_i)|^\tau} \\ & \geq C_{c_0, c_1, \tau}r_k^{\frac{7}{4}\tau^4}, \end{aligned}$$

a contradiction.

Case A.2.3.2. For any $i, j \in \{n_k, n_k + 1, n_k + 2\}$

$$\begin{aligned} q_{i,s_i}m_iM_j &= q_{j,s_j}m_jM_i \\ p_{i,s_i}m_iM_j &= p_{j,s_j}m_jM_i. \end{aligned}$$

Then for $n = n_k$,

$$\frac{p_{n,s_n}}{q_{n,s_n}} = \frac{p_{n+1,s_{n+1}}}{q_{n+1,s_{n+1}}} = \frac{p_{n+2,s_{n+2}}}{q_{n+2,s_{n+2}}}.$$

Hence we can let $p = p_{n,s_n} = p_{n+1,s_{n+1}} = p_{n+2,s_{n+2}}$ and $q = q_{n,s_n} = q_{n+1,s_{n+1}} = q_{n+2,s_{n+2}}$. Then we would have (after plugging in $M_n = ql_{1,n} - pl_{2,n}$)

$$(7.13) \quad q(m_nl_{1,n+1} - m_{n+1}l_{1,n}) = p(m_nl_{2,n+1} - m_{n+1}l_{2,n})$$

$$(7.14) \quad q(m_nl_{1,n+2} - m_{n+2}l_{1,n}) = p(m_nl_{2,n+2} - m_{n+2}l_{2,n})$$

$$(7.15) \quad q(m_{n+1}l_{1,n+2} - m_{n+2}l_{1,n+1}) = p(m_{n+1}l_{2,n+2} - m_{n+2}l_{2,n+1})$$

Then considering $(7.13) \cdot (-l_{1,n+2}) + (7.14) \cdot l_{1,n+1} + (7.15) \cdot (-l_{1,n})$, we get

$$p \cdot \begin{vmatrix} m_{n_k} & l_{1,n_k} & l_{2,n_k} \\ m_{n_k+1} & l_{1,n_k+1} & l_{2,n_k+1} \\ m_{n_k+2} & l_{1,n_k+2} & l_{2,n_k+2} \end{vmatrix} = q \cdot 0 = 0,$$

a contradiction with the choice of n_k .

Case B. $(\alpha_1, \alpha_2) \notin PDC(\tau)$. By the definition of $PDC(\tau)$, the sequence $\vec{h}_n = (h_{1,n}, h_{2,n})$ for which (7.1) fails has to satisfy either $\gcd(h_{1,n}, h_{2,n}) = 1$ (Case B.1) or $h_{1,n}h_{2,n} = 0$ (Case B.2).

Case B.1. We can find a sequence $\{n_j\}$, such that $|\vec{h}_{n_j}| = \max(|h_{1,n_j}|, |h_{2,n_j}|) \rightarrow \infty$ as $j \rightarrow \infty$, $\gcd(h_{1,n_j}, h_{2,n_j}) = 1$ and $\|h_{1,n_j}\alpha_1 + h_{2,n_j}\alpha_2\|_{\mathbb{T}} < \frac{1}{|\vec{h}_{n_j}|^\tau}$.

Without loss of generality, we can assume $|h_{1,n_j}| = |\vec{h}_{n_j}|$. In this case we can take $r_{n_j} = \frac{1}{|h_{1,n_j}|}$. For simplicity we will denote n_j by n .

Now that $\|h_{1,n}\alpha_1 + h_{2,n}\alpha_2\|_{\mathbb{T}} < \frac{1}{|h_{1,n}|^\tau}$, we can find $l_{1,n}, l_{2,n} \in \mathbb{Z}$ such that $|h_{1,n}(\alpha_1 - l_{1,n}) + h_{2,n}(\alpha_2 - l_{2,n})| < \frac{1}{|h_{1,n}|^\tau}$. Since replacing (α_1, α_2) with $(\alpha_1 + l_{1,n}, \alpha_2 + l_{2,n})$ would not change anything, we will assume $|h_{1,n}\alpha_1 + h_{2,n}\alpha_2| < \frac{1}{|h_{1,n}|^\tau}$. Then

$$(7.16) \quad \left| \frac{\alpha_2}{\alpha_1} - \left(-\frac{h_{1,n}}{h_{2,n}}\right) \right| < \frac{1}{|h_{1,n}|^\tau \alpha_1}.$$

We consider the following two lines on \mathbb{T}^2 :

$$l_1(t) = (\{t\}, \{\frac{\alpha_2}{\alpha_1}t\}) \quad \text{and} \quad l_2(t) = (\{t\}, \{-\frac{h_{1,n}}{h_{2,n}}t\}).$$

These two lines are close to each other in the sense that for $|t| \leq |h_{1,n}|^{3\tau/4}$, by (7.16),

$$\left\| \left\{ \frac{\alpha_2}{\alpha_1}t \right\} - \left\{ -\frac{h_{1,n}}{h_{2,n}}t \right\} \right\|_{\mathbb{T}} \leq \left| \frac{\alpha_2}{\alpha_1}t + \frac{h_{1,n}}{h_{2,n}}t \right| \leq \frac{|t|}{|h_{1,n}|^\tau \alpha_1} \leq \frac{1}{|h_{1,n}|^{\tau/4} \alpha_1}.$$

The graph of $l_2(t)$ is the hypotenuse of a right triangle with two legs of lengths $|h_{1,n}|$ and $|h_{2,n}| \pmod{\mathbb{Z}^2}$. We consider the orbit of $(\alpha_1, -\frac{h_{1,n}}{h_{2,n}}\alpha_1)$ under the rotation $(\alpha_1, -\frac{h_{1,n}}{h_{2,n}}\alpha_1)$. These points lie on $l_2(t)$. Under this rotation the point moves a distance $\frac{\sqrt{h_{1,n}^2 + h_{2,n}^2}}{|h_{2,n}|}\alpha_1$ at each step by a big interval with length $\sqrt{h_{1,n}^2 + h_{2,n}^2}$. Let $\{\frac{p_m}{q_m}\}_{m=1}^\infty$ be the continued fraction approximants of $\frac{\alpha_1}{h_{2,n}}$. Choose m such that

$$(7.17) \quad q_{m-1} \leq |h_{1,n}| \sqrt{h_{1,n}^2 + h_{2,n}^2} < q_m.$$

Then it would take a point on \mathbb{T} at most $q_m + q_{m-1}$ steps (under the $\frac{\alpha_1}{h_{2,n}}$ -rotation) to enter each interval of length $\frac{1}{|h_{1,n}| \sqrt{h_{1,n}^2 + h_{2,n}^2}}$ on \mathbb{T} (e.g. [14]), which means it would take a point on $l_2(t)$ at most $q_m + q_{m-1} - 1$ steps (under the $\frac{\sqrt{h_{1,n}^2 + h_{2,n}^2}}{|h_{2,n}|}\alpha_1$ -rotation) to enter each interval of length $\frac{1}{|h_{1,n}|} = r_n$ on the graph of $l_2(t)$. Moreover, it is easy to see that the distance from any $x \in \mathbb{T}^2$ to $l_2(t)$ is bounded by $\frac{1}{\sqrt{h_{1,n}^2 + h_{2,n}^2}} < r_n$. Thus

$$(7.18) \quad \mathbb{T}^2 \subseteq \bigcup_{k=0}^{q_m + q_{m-1}} B_{2r_n}(k\alpha_1, -\frac{h_{1,n}}{h_{2,n}}k\alpha_1).$$

By (2.7) and (7.16),

$$|p_{m-1} + q_{m-1} \frac{\alpha_2}{h_{1,n}}| = |p_{m-1} - q_{m-1} \frac{\alpha_1}{h_{2,n}} + q_{m-1}(\frac{\alpha_1}{h_{2,n}} + \frac{\alpha_2}{h_{1,n}})| \leq \frac{1}{q_m} + \frac{q_{m-1}}{|h_{1,n}|^{\tau-1}}.$$

This implies, by (2.7) and (7.17),

$$\begin{aligned} \|q_{m-1}\alpha_1\|_{\mathbb{T}} &\leq |q_{m-1}\alpha_1 - h_{2,n}p_{m-1}| \leq \frac{|h_{2,n}|}{q_m}, \\ \|q_{m-1}\alpha_2\|_{\mathbb{T}} &\leq \frac{|h_{1,n}|}{q_m} + \frac{2}{|h_{1,n}|^{\tau-4}}. \end{aligned}$$

Then by the fact that $\alpha \in WDC(c_0, \frac{\tau}{4})$ and (7.17),

$$\max\left\{\frac{|h_{2,n}|}{q_m}, \frac{|h_{1,n}|}{q_m} + \frac{2}{|h_{1,n}|^{\tau-4}}\right\} \geq \max(\|q_{m-1}\alpha_1\|_{\mathbb{T}}, \|q_{m-1}\alpha_2\|_{\mathbb{T}}) \geq \frac{c_0}{q_{m-1}^{\tau/4}} \geq \frac{c_0}{2^{\frac{\tau}{4}}|h_{1,n}|^{\tau/2}}.$$

This implies

$$(7.19) \quad q_m + q_{m-1} < 2q_m \leq \frac{2^{\frac{\tau}{4}+2}}{c_0}|h_{1,n}|^{\tau/2+1}.$$

Since $0 \leq k \leq \frac{2^{\frac{\tau}{4}+2}}{c_0}|h_{1,n}|^{\tau/2+1} < r_n^{-\frac{3\tau}{4}}$, by (7.16) the points $(k\alpha_1, k\alpha_2)$ and $(k\alpha_1, -\frac{h_{1,n}}{h_{2,n}}k\alpha_1)$ differ at most by $r_n^{\frac{\tau}{4}}$, so we obtain using (7.18) and (7.19),

$$\mathbb{T}^2 \subseteq \cup_{k=0}^{r_n^{-3\tau/4}} B_{3r_n}(k\alpha_1, k\alpha_2).$$

Case B.2. We can find a sequence $\{n_j\}$ such that $h_{2,n_j} \equiv 0$ and $|h_{1,n_j}| \rightarrow \infty$ such that

$$(7.20) \quad \|h_{1,n_j}\alpha_1\|_{\mathbb{T}} < \frac{1}{|h_{1,n_j}|^{\tau}}.$$

For simplicity we will replace n_j with n . We can find M_n such that $|h_{1,n}\alpha_1 - M_n| < \frac{1}{|h_{1,n}|^{\tau}}$. Let $d_n = \gcd(h_{1,n}, M_n)$ be the greatest common divisor. Let $\tilde{h}_{1,n} = \frac{h_{1,n}}{d_n}$ and $\tilde{M}_n = \frac{M_n}{d_n}$. We have

$$(7.21) \quad |\alpha_1 - \frac{\tilde{M}_n}{\tilde{h}_{1,n}}| < \frac{1}{|h_{1,n}|^{\tau+1}} \rightarrow 0.$$

If $\tilde{h}_{1,n}$ is bounded in n , then α_1 can be approximated arbitrarily closely by rationals with bounded denominators, which is impossible. Thus $|\tilde{h}_{1,n}| \rightarrow \infty$. Now take radius $r_n = \frac{1}{|\tilde{h}_{1,n}|}$. For each $0 \leq i \leq \tilde{h}_{1,n} - 1$ consider $\{(i\alpha_1 + k\tilde{h}_{1,n}\alpha_1, i\alpha_2 + k\tilde{h}_{1,n}\alpha_2)\}_{k=0}^{\infty}$. Let $\{\frac{p_m}{q_m}\}_{m=1}^{\infty}$ be the continued fraction approximants of $\tilde{h}_{1,n}\alpha_2$. Choose m such that

$$(7.22) \quad q_{m-1} \leq |\tilde{h}_{1,n}| = r_n^{-1} < q_m.$$

Then it takes any point on \mathbb{T} at most $q_m + q_{m-1} - 1$ steps (under the $\tilde{h}_{1,n}\alpha_2$ -rotation) to enter each interval of length r_n [14]. By (2.7),

$$(7.23) \quad |p_{m-1} - q_{m-1}\tilde{h}_{1,n}\alpha_2| \leq \frac{1}{q_m}.$$

By (7.20), (7.22) and since $\tau > 4$, we have $\|q_{m-1}\tilde{h}_{1,n}\alpha_1\| \leq \frac{q_{m-1}}{|\tilde{h}_{1,n}|^{\tau}} < \frac{c_0}{(q_{m-1}|\tilde{h}_{1,n}|)^{\tau/4}}$. By the fact that $\alpha \in WDC(c_0, \frac{\tau}{4})$, $\|q_{m-1}\tilde{h}_{1,n}\alpha_2\| \geq \frac{c_0}{(q_{m-1}|\tilde{h}_{1,n}|)^{\tau/4}}$. By (7.23) and (7.22), we have

$$(7.24) \quad q_m \leq \frac{1}{c_0}|\tilde{h}_{1,n}|^{\frac{\tau}{2}}.$$

Now for $0 \leq k \leq q_m + q_{m-1} - 1$, by (7.21), (7.20) and (7.24), $\|i\alpha_1 + k\tilde{h}_{1,n}\alpha_1 - \frac{i\tilde{M}_n}{\tilde{h}_{1,n}}\|_{\mathbb{T}} \leq \frac{C}{|\tilde{h}_{1,n}|^{\frac{\tau}{2}}} = Cr_n^{\frac{\tau}{2}}$. Since $\gcd(\tilde{h}_{1,n}, \tilde{M}_n) = 1$, any interval of length $r_n = \frac{1}{|\tilde{h}_{1,n}|}$ contains $\frac{i\tilde{M}_n}{\tilde{h}_{1,n}}$ for some $0 \leq i \leq \tilde{h}_{1,n} - 1$. Thus

$$\mathbb{T}^2 \subseteq \cup_{k=0}^{(q_m+q_{m-1})|\tilde{h}_{1,n}|} B_{r_n}(k\alpha_1, k\alpha_2).$$

By (7.24), $(q_m + q_{m-1})|\tilde{h}_{1,n}| \leq r_n^{-\tau}$, so we have

$$(7.25) \quad \mathbb{T}^2 \subseteq \cup_{k=0}^{r_n^{-\tau}} B_{r_n}(k\alpha_1, k\alpha_2).$$

□.

APPENDIX A

Proof of Lemma 3.6. We include the proof here for completeness.

For sufficiently small $\epsilon > 0$, fix an integer $H_0 \sim N^{1/(d(\tau-1)+1+d\epsilon)}$, define $g(n) = \frac{1}{n(n+1)}$ for $1 \leq n < H_0$ and $g(H_0) = \frac{1}{H_0}$. For $(n_1, \dots, n_d) \in \mathbb{Z}^d$ with $1 \leq n_i \leq H_0$, define $f(n_1, \dots, n_d) = \prod_{i=1}^d g(n_i)$. By Lemma 2.2, we have

$$\begin{aligned}
D_N(\theta) &\leq C_d \left(\frac{1}{H_0} + \sum_{0 < |h| \leq H_0} \frac{1}{r(\vec{h})} \left| \frac{1}{N} \sum_{n=1}^N e^{2\pi i \langle \vec{h}, \alpha \rangle n} \right| \right) \\
&\leq \tilde{C}_d \left(\frac{1}{H_0} + \frac{1}{N} \sum_{0 < |h| \leq H_0} \frac{1}{r(\vec{h})} \frac{1}{\|\langle \vec{h}, \alpha \rangle\|_{\mathbb{T}}} \right) \\
&= \tilde{C}_d \left(\frac{1}{H_0} + \frac{1}{N} \sum_{n_1, \dots, n_d=1}^{H_0} f(n_1, \dots, n_d) \sum_{\vec{h}=(h_1, \dots, h_d) \neq \vec{0}, |h_j| \leq n_j} \frac{1}{\|\langle \vec{h}, \alpha \rangle\|_{\mathbb{T}}} \right) \\
&\leq \tilde{C}_d \left(\frac{1}{H_0} + \frac{1}{N} \sum_{n_1, \dots, n_d=1}^{H_0} f(n_1, \dots, n_d) \sum_{j=1}^{3^d r(\vec{n})} \frac{r(\vec{n})^\tau}{j} \right) \\
&\leq \tilde{C}_d \left(\frac{1}{H_0} + \frac{1}{N} \sum_{n_1, \dots, n_d=1}^{H_0} f(n_1, \dots, n_d) r(\vec{n})^\tau \log r(\vec{n}) \right) \\
&\leq \tilde{C}_d \left(\frac{1}{H_0} + \frac{H_0^{d(\tau-1+\epsilon)}}{N} \right) \\
&\lesssim N^{-1/(d(\tau-1)+1+d\epsilon)}.
\end{aligned}$$

□

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